Using Integrating Factor Method to Solve Some Types of Fractional Differential Equations

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***Abstract*—This paper uses integrating factor method, product rule and chain rule for fractional derivatives to find the general solutions of some types of first order fractional differential equations, regarding Jumarie's modified Riemann-Liouville (R-L) fractional derivative. Moreover, an example is proposed to illustrate our result.**

***Index Terms*—Integrating factor method, Product rule, Chain rule, Jumarie's modified R-L fractional derivative.**

I. INTRODUCTION

The derivative of non-integer order has been an interesting research topic for several centuries. The idea was motivated by the question, " What does it mean by $\frac{d^{1/2}}{dx^{1/2}}f(x)$? ", asked by L'Hospital in 1695 in his letters to Leibniz [1-3]. Since then, the mathematicians tried to answer this question for centuries in several points of view. The outcomes are many folds. Many various types of fractional derivatives were introduced: Riemann-Liouville, Caputo, Hadamard, Erd´elyi-Kober, Gr¨unwald-Letnikov, Marchaud and Riesz are just a few to name [4-6].

Fractional differential equations can describe the dynamics of several complex and nonlocal systems with memory. They arise in many scientific and engineering areas such as physics, chemistry, biology, biophysics, economics, control theory, signal and image processing, etc [5-10]. The general solution of exact fractional differential equation has been obtain [11]. In this article, we study some types of non-exact fractional differential equation, regarding the Jumarie type of modified Riemann-Liouville (R-L) fractional derivatives. We define a new multiplication of fractional functions and use the integrating factor method, the product rule and chain rule for fractional derivatives to find the general solutions of these special fractional differential equations. In fact, our results are generalizations of classical ordinary differential equations. Furthermore, an example is given to demonstrate the advantage of our result. In addition, the papers study on fractional differential equations can refer to [11-14].
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II. PRELIMINARIES AND METHODS

At first, we introduce the fractional calculus used in this paper.

Definition 2.1: Suppose that α is a real number and m is a positive integer.Then the modified R-L fractional derivative of Jumarie type is defined by ([15])

$$
{}_aD_x^{\alpha}[f(x)] = \begin{cases} \dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_a^x (x-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha<0 \\ \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_a^x (x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0\le\alpha<1 \\ \dfrac{d^m}{dx^m}\left({}_aD_x^{\alpha-m}\right)[f(x)], & \text{if } m\le\alpha<m+1 \end{cases} \tag{1}
$$

where$\Gamma(y)=\int_0^{\infty} t^{y-1}e^{-t}dt$ is the gamma function defined on $y>0$. If $\left({}_aD_x^{\alpha}\right)^n[f(x)] = \left({}_aD_x^{\alpha}\right)\left({}_aD_x^{\alpha}\right)\cdots\left({}_aD_x^{\alpha}\right)[f(x)]$ exists, then $f(x)$ is called n-th order α-fractional differentiable function, and $\left({}_aD_x^{\alpha}\right)^n[f(x)]$ is the n-th order α-fractionalderivative of $f(x)$. On the other hand, we definethe α -fractional integral of $f(x)$, ${}_aI_x^{\alpha}[f(x)] = {}_aD_x^{-\alpha}[f(x)]$, where$\alpha>0$, and $f(x)$ is called α-fractional integral function. Furthermore, if $M_\alpha(x^\alpha,y^\alpha)$ is a two-variable α-fractional function defined on$[a,b]\times[c,d]$, then we define ${}_a\partial_x^{\alpha}[M_\alpha(x^\alpha,y^\alpha)]$ and ${}_c\partial_y^{\alpha}[M_\alpha(x^\alpha,y^\alpha)]$ are α-fractional partial derivatives with respect to x and y respectively. And, ${}_aJ_x^{\alpha}[M_\alpha(x^\alpha,y^\alpha)]$ and ${}_cJ_y^{\alpha}[M_\alpha(x^\alpha,y^\alpha)]$ are α -fractional integrals with respect to$x$and $y$respectively.

Proposition 2.2: *If α,β,c are real constants and $0<\alpha\le 1$, then*

$$ {}_0D_x^{\alpha}[x^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha},\ if\beta\ge\alpha \tag{2}$$

$$ {}_0D_x^{\alpha}[c] = 0, \tag{3}$$

and

$$ \left({}_0I_x^{\alpha}\right)[x^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}x^{\beta+\alpha},\quad if\beta>-1. \tag{4}$$

Definition 2.3 ([17])**:** TheMittag-Leffler function is defined by

$$ E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \tag{5}$$

whereα is a real number, $\alpha>0$, and z is a complex variable.

Definition 2.4 ([14])**:** $E_\alpha(\lambda x^\alpha)$ is called α-order fractional exponential function. The α-order fractional cosine and sine function are defined as follows:

$$ cos_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k}x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{6}$$

and

$$ sin_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k+1}x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \tag{7}$$

where $0<\alpha\le 1$, λ is a complex number, and x is a

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realvariable.

Next, we define a new multiplication of fractional functions.

Definition 2.5 ([16])**:** Let λ, μ, z be complex numbers, $0 < \alpha \leq 1$, j, l, k be non-negative integers, and a_k, b_k be real numbers, $p_k(z) = \frac{1}{\Gamma(k\alpha+1)} z^k$ for all k. The $\otimes$ multiplication is defined by

$$p_j(\lambda x^\alpha) \otimes p_l(\mu y^\alpha)$$
$$= \frac{1}{\Gamma(j\alpha+1)}(\lambda x^\alpha)^j \otimes \frac{1}{\Gamma(l\alpha+1)}(\mu y^\alpha)^l$$
$$= \frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(\lambda x^\alpha)^j(\mu y^\alpha)^l, \quad (8)$$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!\,l!}$.

If $f_\alpha(\lambda x^\alpha)$ and $g_\alpha(\mu y^\alpha)$ are two fractional functions,

$$f_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(\lambda x^\alpha)^k, \quad (9)$$

$$g_\alpha(\mu y^\alpha) = \sum_{k=0}^{\infty} b_k\, p_k(\mu y^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(\mu y^\alpha)^k, \quad (10)$$

then we define

$$f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha)$$
$$= \sum_{k=0}^{\infty} a_k\, p_k(\lambda x^\alpha) \otimes \sum_{k=0}^{\infty} b_k\, p_k(\mu y^\alpha)$$
$$= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m} b_m p_{k-m}(\lambda x^\alpha) \otimes p_m(\mu y^\alpha)\right). \quad (11)$$

Proposition 2.6: $f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu y^\alpha)$

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m (\lambda x^\alpha)^{k-m} (\mu y^\alpha)^m. \quad (12)$$

Definition 2.7: Let $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes n} = f_\alpha(\lambda x^\alpha) \otimes \cdots \otimes f_\alpha(\lambda x^\alpha)$ be the n times product of the fractional function $f_\alpha(\lambda x^\alpha)$. If $f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\lambda x^\alpha) = 1$, then $g_\alpha(\lambda x^\alpha)$ is called the $\otimes$ reciprocal of $f_\alpha(\lambda x^\alpha)$, and is denoted by $\left(f_\alpha(\lambda x^\alpha)\right)^{\otimes -1}$.

Theorem 2.8 (product rule for fractional derivatives) ([16])**:** *If $0 < \alpha \leq 1$, λ, μ are complex numbers, and f_α, g_α are fractional functions. Then*

$$\left({}_0D_x^\alpha\right)[f_\alpha(\lambda x^\alpha) \otimes g_\alpha(\mu x^\alpha)]$$
$$= \left({}_0D_x^\alpha\right)[f_\alpha(\lambda x^\alpha)] \otimes g_\alpha(\mu x^\alpha) + f_\alpha(\lambda x^\alpha) \otimes \left({}_0D_x^\alpha\right)[g_\alpha(\mu x^\alpha)]. \quad (13)$$

Theorem 2.9 (chain rule for fractional derivatives) ([16])**:** *If* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g_\alpha(\mu x^\alpha) = \sum_{k=0}^{\infty} b_k\, p_k(\mu x^\alpha)$. *Let* $f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{k=0}^{\infty} a_k \left(g_\alpha(\mu x^\alpha)\right)^{\otimes k}$, $f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) = \sum_{k=1}^{\infty} a_k\, k\left(g_\alpha(\mu x^\alpha)\right)^{\otimes(k-1)}$, *then*

$$\left({}_0D_x^\alpha\right)[f_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right)] = f'_{\otimes\alpha}\left(g_\alpha(\mu x^\alpha)\right) \otimes \left({}_0D_x^\alpha\right)[g_\alpha(\mu x^\alpha)]. \quad (14)$$

Definition 2.10: Let x, y be real variables, $y = y(x): [a,b] \to [c,d]$, $y(a) = c$, $0 < \alpha \leq 1$. Suppose that $M_\alpha(x^\alpha, y^\alpha)$, $N_\alpha(x^\alpha, y^\alpha)$ defined on $[a,b] \times [c,d]$, and have continuous first-order α-fractional partial derivatives, $N_\alpha(x^\alpha, y^\alpha) \neq 0$, then

$${}_aD_x^\alpha[y(x^\alpha)] = -M_\alpha(x^\alpha, y^\alpha) \otimes \left(N_\alpha(x^\alpha, y^\alpha)\right)^{\otimes -1} \quad (15)$$

is called exact α-fractional differential equation, if

$${}_c\partial_y^\alpha[M_\alpha(x^\alpha, y^\alpha)] = {}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)].$$

Theorem 2.11: *Let the assumptions be the same as Definition 2.10, and C be a constant. Then the α-fractional differential equation*

$${}_aD_x^\alpha[y(x^\alpha)] = -M_\alpha(x^\alpha, y^\alpha) \otimes \left(N_\alpha(x^\alpha, y^\alpha)\right)^{\otimes -1}$$

is exact if and only if it has the general solution

$$F(x^\alpha, y^\alpha) = {}_aJ_x^\alpha[M_\alpha(x^\alpha, y^\alpha)] + {}_cJ_y^\alpha[N_\alpha(a, y^\alpha)] = C. \quad (16)$$

Proof *Only if part* **:** Let $F(x^\alpha, y^\alpha)$ be two variables α-fractional function such that

$$\begin{cases} {}_a\partial_x^\alpha[F(x^\alpha, y^\alpha)] = M_\alpha(x^\alpha, y^\alpha) \\ {}_c\partial_y^\alpha[F(x^\alpha, y^\alpha)] = N_\alpha(x^\alpha, y^\alpha). \end{cases} \quad (17)$$

Then $F(x^\alpha, y^\alpha) = {}_aJ_x^\alpha[M_\alpha(x^\alpha, y^\alpha)] + g(y^\alpha)$ for some α-fractional function $g(y^\alpha)$, and hence

$${}_c\partial_y^\alpha[F(x^\alpha, y^\alpha)] = {}_aJ_x^\alpha\left[{}_c\partial_y^\alpha[M_\alpha(x^\alpha, y^\alpha)]\right] + g'(y^\alpha). \quad (18)$$

Since ${}_c\partial_y^\alpha[M_\alpha(x^\alpha, y^\alpha)] = {}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)]$, it follows that

$${}_c\partial_y^\alpha[F(x^\alpha, y^\alpha)] = {}_aJ_x^\alpha\left[{}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)]\right] + g'(y^\alpha)$$
$$= N_\alpha(x^\alpha, y^\alpha) - N_\alpha(a, y^\alpha) + g'(y^\alpha). \quad (19)$$

Thus, $g'(y^\alpha) = N_\alpha(a, y^\alpha)$. Take $g(y^\alpha) = {}_cJ_y^\alpha[N_\alpha(a, y^\alpha)]$, then

$$F(x^\alpha, y^\alpha) = {}_aJ_x^\alpha[M_\alpha(x^\alpha, y^\alpha)] + {}_cJ_y^\alpha[N_\alpha(a, y^\alpha)]. \quad (20)$$

Therefore, if $F(x^\alpha, y^\alpha) = C$ determines a α-fractional implicit function $y = y(x^\alpha)$, i.e.,

$$F\left(x^\alpha, y(x^\alpha)\right) = C. \quad (21)$$

Then fractional differentiating x on both sides of Eq. (21), we can obtain

$${}_a\partial_x^\alpha[F(x^\alpha, y^\alpha)] + {}_c\partial_y^\alpha[F(x^\alpha, y^\alpha)] \otimes {}_aD_x^\alpha[y(x^\alpha)] = 0. \quad (22)$$

And hence, Eq. (15) holds. This means the implicit function $y = y(x^\alpha)$ determined by Eq. (21) is the solution of Eq. (15). Thus, $F(x^\alpha, y^\alpha) = C$ is the general solution of Eq. (15).

If part **:** Using Eqs. (17) yields

$${}_c\partial_y^\alpha\left[{}_a\partial_x^\alpha[F(x^\alpha, y^\alpha)]\right] = {}_c\partial_y^\alpha[M_\alpha(x^\alpha, y^\alpha)], \quad (23)$$

and

$${}_a\partial_x^\alpha\left[{}_c\partial_y^\alpha[F(x^\alpha, y^\alpha)]\right] = {}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)]. \quad (24)$$

Since ${}_c\partial_y^\alpha\left[{}_a\partial_x^\alpha[F(x^\alpha, y^\alpha)]\right] = {}_a\partial_x^\alpha\left[{}_c\partial_y^\alpha[F(x^\alpha, y^\alpha)]\right]$, it follows that

$${}_c\partial_y^\alpha[M_\alpha(x^\alpha, y^\alpha)] = {}_a\partial_x^\alpha[N_\alpha(x^\alpha, y^\alpha)].$$

Q.e.d.

Definition 2.12: If there exists a α-fractional function $\mu_\alpha(x^\alpha, y^\alpha) \neq 0$ such that

$${}_c\partial_y^\alpha[\mu_\alpha(x^\alpha, y^\alpha) \otimes M_\alpha(x^\alpha, y^\alpha)]$$
$$= {}_a\partial_x^\alpha[\mu_\alpha(x^\alpha, y^\alpha) \otimes N_\alpha(x^\alpha, y^\alpha)]. \quad (25)$$

Then $\mu_\alpha(x^\alpha, y^\alpha)$ is called the α-fractional integrating factor of Eq. (15).

III. Result And Example

The following is the main result in this paper.

Theorem 3.1: *Suppose that the assumptions are the same* *as Definition 2.10.*

Case 1. If there exists a α-fractional integrating factor $\mu_\alpha(x^\alpha)$*of Eq. (15), then*

$$\left({}_c\partial_y^\alpha[M_\alpha(x^\alpha,y^\alpha)] - {}_a\partial_x^\alpha[N_\alpha(x^\alpha,y^\alpha)]\right) \otimes \left(N_\alpha(x^\alpha,y^\alpha)\right)^{\otimes -1} = \psi(x^\alpha), (26)$$

and

$$\mu_\alpha(x^\alpha) = E_\alpha\left(\left({}_aI_x^\alpha\right)[\psi(x^\alpha)]\right). \quad (27)$$

Case 2. If there is a α-fractional integrating factor $\mu_\alpha(y^\alpha)$*of Eq. (15), then*

$$\left({}_c\partial_y^\alpha[M_\alpha(x^\alpha,y^\alpha)] - {}_a\partial_x^\alpha[N_\alpha(x^\alpha,y^\alpha)]\right) \otimes -\left(M_\alpha(x^\alpha,y^\alpha)\right)^{\otimes -1} = \varphi(y^\alpha), (28)$$

and

$$\mu_\alpha(y^\alpha) = E_\alpha\left(\left({}_cI_y^\alpha\right)[\varphi(y^\alpha)]\right). \quad (29)$$

Proof *Case 1.* Since

$${}_c\partial_y^\alpha[\mu_\alpha(x^\alpha) \otimes M_\alpha(x^\alpha,y^\alpha)] = {}_a\partial_x^\alpha[\mu_\alpha(x^\alpha) \otimes N_\alpha(x^\alpha,y^\alpha)],$$

it follows from product rule for fractional derivatives that

$$\mu_\alpha(x^\alpha) \otimes {}_c\partial_y^\alpha[M_\alpha(x^\alpha,y^\alpha)] = {}_aD_x^\alpha[\mu_\alpha(x^\alpha)] \otimes N_\alpha(x^\alpha,y^\alpha) + \mu_\alpha(x^\alpha) \otimes {}_a\partial_x^\alpha[N_\alpha(x^\alpha,y^\alpha)]. \quad (30)$$

And hence,

$$\mu_\alpha(x^\alpha) \otimes \left({}_c\partial_y^\alpha[M_\alpha(x^\alpha,y^\alpha)] - {}_a\partial_x^\alpha[N_\alpha(x^\alpha,y^\alpha)]\right) = {}_aD_x^\alpha[\mu_\alpha(x^\alpha)] \otimes N_\alpha(x^\alpha,y^\alpha). (31)$$

Therefore,

$$\left(\mu_\alpha(x^\alpha)\right)^{\otimes -1} \otimes {}_aD_x^\alpha[\mu_\alpha(x^\alpha)] = \left({}_c\partial_y^\alpha[M_\alpha(x^\alpha,y^\alpha)] - {}_a\partial_x^\alpha[N_\alpha(x^\alpha,y^\alpha)]\right) \otimes \left(N_\alpha(x^\alpha,y^\alpha)\right)^{\otimes -1}. \quad (32)$$

So,

$$\left({}_c\partial_y^\alpha[M_\alpha(x^\alpha,y^\alpha)] - {}_a\partial_x^\alpha[N_\alpha(x^\alpha,y^\alpha)]\right) \otimes \left(N_\alpha(x^\alpha,y^\alpha)\right)^{\otimes -1} = \psi(x^\alpha)$$

for some α-fractional function $\psi(x^\alpha)$. And

$${}_aD_x^\alpha[\mu_\alpha(x^\alpha)] = \mu_\alpha(x^\alpha) \otimes \psi(x^\alpha). \quad (33)$$

On the other hand, by chain rule for fractional derivatives, we obtain

$${}_aD_x^\alpha\left[E_\alpha\left(\left({}_aI_x^\alpha\right)[\psi(x^\alpha)]\right)\right] = E_\alpha\left(\left({}_aI_x^\alpha\right)[\psi(x^\alpha)]\right) \otimes \psi(x^\alpha),$$

i.e., $E_\alpha\left(\left({}_aI_x^\alpha\right)[\psi(x^\alpha)]\right)$satisfies the first order α-fractional differential equation Eq. (33). And hence, $\mu_\alpha(x^\alpha) = E_\alpha\left(\left({}_aI_x^\alpha\right)[\psi(x^\alpha)]\right)$.

Case 2. Using the same method as *Case 1*, we can easily obtain the desired result.Q.e.d.

Example 3.2: Considerthe first order $1/2$-fractional differential equation

$${}_0D_x^{1/2}\left[y\left(x^{1/2}\right)\right] = cos_{1/2}\left(x^{1/2}\right) \otimes y\left(x^{1/2}\right) + sin_{1/2}\left(x^{1/2}\right). \quad (34)$$

Since the$1/2$-fractional integrating factor of Eq. (34) is

$$\mu_{1/2}\left(x^{1/2}\right) = E_{1/2}\left(-\left({}_0I_x^{1/2}\right)\left[cos_{1/2}\left(x^{1/2}\right)\right]\right) = E_{1/2}\left(-sin_{1/2}\left(x^{1/2}\right)\right). \quad (35)$$

Then multiply $E_{1/2}\left(-sin_{1/2}\left(x^{1/2}\right)\right)$on both sides of Eq. (34), we have

$$E_{1/2}\left(-sin_{1/2}\left(x^{1/2}\right)\right) \otimes \left(cos_{1/2}\left(x^{1/2}\right) \otimes y\left(x^{1/2}\right) + \right.$$
sin12x12−0Dx12yx12=0. (36)

Therefore,

$$-{}_0D_x^{1/2}\left[E_{1/2}\left(-sin_{1/2}\left(x^{1/2}\right)\right) \otimes y\left(x^{1/2}\right)\right] + {}_0D_x^{1/2}\left[\left({}_0I_x^{1/2}\right)\left[sin_{1/2}\left(x^{1/2}\right) \otimes E_{1/2}\left(-sin_{1/2}\left(x^{1/2}\right)\right)\right]\right] = 0. \quad (37)$$

And hence, the general solution of Eq. (34) is

$$F\left(x^{1/2}, y^{1/2}\right) = \left({}_0I_x^{1/2}\right)\left[sin_{1/2}\left(x^{1/2}\right) \otimes E_{1/2}\left(-sin_{1/2}\left(x^{1/2}\right)\right) - \right.$$
E12−sin12x12⊗yx12=C. (38)

Q.e.d.

IV. Conclusion

As mentioned above, we can obtain the general solution of some types of non-exact fractional differential equations by using integrating factor method, product rule and chain rule for fractional derivatives. Furthermore, our result is the generalization of some types of classical ordinary differential equations. On the other hand, the new multiplication we defined is a natural operation in fractional calculus. In the future, we will use the Jumarie type of modified R-L fractional derivatives and the new multiplication to research the topics on problems of applied science and fractional calculus.

References

[1] G. W. Leibniz, Letter from Hanover, Germany to G.F.A. L'Hospital, September 30, 1695, Leibniz MathematischeSchriften, Olms-Verlag, Hildesheim, Germany, 1962, pp.301-302, First published in 1849.
[2] G. W. Leibniz, Letter from Hanover, Germany to Johann Bernoulli, December 28, 1695, Leibniz MathematischeSchriften, Olms-Verlag, Hildesheim, Germany, 1962, p.226, First published in 1849.
[3] G. W. Leibniz, Letter from Hanover, Germany to John Wallis, May 28, 1697, Leibniz MathematischeSchriften, Olms-Verlag, Hildesheim, Germany, 1962, p.25, First published in 1849.
[4] K. B. Oldham, and J. Spanier, The fractional calculus, Academic Press, New York, 1974.
[5] I. Podlubny, Fractional differential equations, Academic Press, San Diego, California, U.S.A., 1999.
[6] S. G. Samko, A.A. Kilbas, and O. I. Marichev, Fractional integrals and derivatives: theory and applications, Gordon and Breach, 1993.
[7] R. Hilfer, Applications of fractional calculus in physics. World Scientific, New Jersey.
[8] V. E. Tarasov, Fractional dynamics applications of fractional calculus to dynamics of particles. Fields and Media, Higher Education Press, Beijing and Springer-Verlag, Berlin Heidelberg, 2010.
[9] G. M. Zaslavsky, Hamiltonian chaos and fractional dynamics, Oxford University Press, Oxford, 2005.
[10] A. Carpinteri, and F. Mainardi (Eds.), Fractals and fractional calculus in continuum mechanics, Springer, Wien, 1997.
[11] C. -H. Yu, A study of exact fractional differential equations, International Journal of Interdisciplinary Research and Innovations, Vol. 8, Issue 4, pp. 100-105, 2020.
[12] C. -H. Yu, Separable fractional differential equations, International Journal of Mathematics and Physical Sciences Research, Vol. 8, Issue 2, pp. 30-34, 2020.
[13] C. -H. Yu, Integral form of particular solution of nonhomogeneous linear fractional differential equation with constant coefficients,

International Journal of Novel Research in Engineering and Science, Vol. 7, Issue 2, pp. 1-9, 2020.
[14] C. -H. Yu, Fractional derivatives of some fractional functions and their applications, Asian Journal of Applied Science and Technology, Vol. 4, Issue 1, pp. 147-158, 2020.
[15] D. Kumar, and J. Daiya, Linear fractional non-homogeneous differential equations with Jumarie fractional derivative, Journal of Chemical, Biological and Physical Sciences, Vol. 6, No. 2, pp. 607-618, 2016.
[16] C. -H. Yu, Differential properties of fractional functions, International Journal of Novel Research in Interdisciplinary Studies, Vol. 7, No. 5, pp. 1-14, 2020.
[17] J. C. Prajapati , Certain properties of Mittag-Leffler function with argument x^{α}, $\alpha > 0$. Italian Journal of Pure and Applied Mathematics, Vol. 30, pp. 411-416, 2013.

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