ISSN: 2454-8236, Volume-9, Issue-3, September 2020 Pages 110-114

On The Stability of Order 7 Rational Interpolation Scheme for Solving Initial Value Problems in Ordinary Differential Equation

Ehiemua M. E., Agbeboh G.U.

***Abstract*- The validity of a method can be verified by analyzing the stability property of a rational interpolation scheme. In this article, investigation is carried out on a rational interpolation scheme of order 7, through two major processes of converting the resulting rational function to a complex outlook, and then, transforming the complex function to a polar form, from which the stability region of the method is constructed in the form of Jordan Curve. The regions of stability and instability as well as the encroachment interval of the scheme is determined with the use of Maple-18 and Matlab packages**

***Index Terms*- Rational Interpolation, Consistency, A-Stability, Region of Instability, Region of Stability, Encroachment interval**

.

I. INTRODUCTION

Numerical methods for evaluating systems of Ordinary Differential Equations (ODEs) have been attracting much attention because they proffer the solutions of problems arising from the mathematical formulation of physical situations such as those in chemical kinetics, population, economic, political and social models. Numerical solution of ordinary differential equations can be obtained using rational integrators, such as linear multistep, Runge-Kutta and exponential methods and many others.

The object of our study is the stability function of a general rational integrator reported in Aashikpelokhai [1] and whose underlying interpolant is a rational function $P_L(x)Q_M(x)^{-1}$ where $P_L(x)$ and $Q_M(x)$ are polynomial functions of degrees L and M respectively.

The stability function of any integrator is what is normally used in determining the Region of absolute stability of such an integrator. A common yardstick which is used in the determination of the Region of Absolute Stability (RAS) is the unit ball in R^n .. The stability function $S(\bar{h})$ is defined as the ratio $\frac{y_{n+1}}{y_n}$

Ehiemua M. E., Department of Mathematics, Ambrose Alli University, Ekpoma, Edo State. Nigeria.
Agbeboh G. U., Department of Mathematics, Ambrose Alli University, Ekpoma, Edo State. Nigeria.

Where $\frac{y_{n+1}}{y_n} = S(\bar{h}) = R_{L,M}(\bar{h})$

and

$\left|S(\bar{h})\right| < 1 \Leftrightarrow \left|R_{L,M}(\bar{h})\right| < 1.$

According to Aashikpelokhai and Elakhe [1], it was shown that $\left|R_{L,M}(\bar{h})\right|$ is A-Stable, if and only if $M-2 \le L \le M$. Consequently, we concentrate our attention on these constrained values of L and M if we have to achieve A-Stable method.

The works of Aashikpelokhai [2] and Agbeboh [3] established that rational integrator methods, can solve problems that are either stiff or non-stiff in nature. Agbeboh and Aashikpelokhai [4] worked on implementation of rational integrator, of order 26, which produced result that compared favorably well with other existing methods without really establishing the stability region of the method. All of these works were basically for the improvement of numerical solutions to initial value problems, which sets the foundation for the study. The region of absolute stability (RAS), is the same as the region in which the absolute value of the stability function lies in the unit circle. As Aashikpelokhai [2] puts it, the smallness of the individual error is called accuracy but the ability to keep the effect of this error under control is called stability. He further opined that the region of absolute stability (RAS) of rational interpolation methods always lie entirely on the left-half of the complex plane. Therefore, for a rational interpolation method to be useful, it should possess an appropriate stability property.

The overall aim of this study is to determine the stability polynomial, and establish the region of absolute stability of a rational integrator of order 7; that is when k = 4. Our computational experience as exemplified by the works of Aashikpelokhai [2], Fatunla and Aashikpelokhai [5], along with the research work given by Fatunla [6], Lambert and Shaw [7], Otunta and Ikhile [8], all give credence to the need for rational integrators. To go about this there is need to first establish the basic matrix of the method, which will give an investigative advantage to obtain result easily. The Crammer's Rule is implemented to enable the determination of the stability function of the method. Basically, there are two ways of approaching the expansion of the stability function; these will be through direct algebraic method using F (u, v) and polar method of F(R, ϕ). By employing the binomial expansion on F (u, v), we established the complex integrating function and reduce the expansion to polar form at which point we introduce F(R, ϕ), where R represents the polynomial of the complex integrator function, while argument (ϕ) is the angle of rotation, about which the roots of R, are determined. After the expansion, the integrator is subjected to a sequence of solvability analysis to establish the stability function for the rational integrator for the case k = 4. The roots of the complex polynomial are determined to enable the plotting of the Jordan curve, which will show both the Region of Absolute stability (RAS) and the Region of instability (RIS).

II. STABILITY ANALYISIS OF THE METHOD

The general rational integrator formula for k = 4, is defined by:

$$U(x)=\frac{\sum_{r=0}^{3}p_r x^r}{1+\sum_{r=1}^{4}q_r x^r} \qquad (2.1)$$

and if given that:

$$U(x)=\sum_{r=0}^{\infty}c_r x^r \qquad (2.2)$$

Therefore (2.1) can be written as:

$$\sum_{r=0}^{\infty}c_r x^r=\frac{\sum_{r=0}^{3}p_r x^r}{1+\sum_{r=1}^{4}q_r x^r} \qquad (2.3)$$

$$\sum_{r=0}^{7}c_r x^r\left(1+\sum_{r=1}^{3}q_r x^r\right)=\sum_{r=0}^{4}p_r x^r \qquad (2.4)$$

Expanding and collecting equal coefficients of $x^{r's}$, we have

$$\begin{aligned}
p_0 &:= c_0\\
p_1 &:= (c_0 q_1 + c_1)\\
p_2 &:= (c_0 q_2 + c_1 q_1 + c_2)\\
p_3 &:= (c_0 q_3 + c_1 q_2 + c_2 q_1 + c_3)\\
p_4 &:= (c_0 q_4 + c_1 q_3 + c_2 q_2 + c_3 q_1 + c_4)\\
p_5 &:= (c_1 q_4 + c_2 q_3 + c_3 q_2 + c_4 q_1 + c_5)\\
p_6 &:= (c_2 q_4 + c_3 q_3 + c_4 q_2 + c_5 q_1 + c_6)\\
p_7 &:= (c_3 q_4 + c_4 q_3 + c_5 q_2 + c_6 q_1 + c_7)
\end{aligned} \qquad (2.5)$$

Where

Since the numerator is order 3, the integrator parameters:

$p_4=p_5=p_6=p_7=p_8=p_9=p_{10}=0$, Hence

$$\left.\begin{aligned}
c_6q_1+c_5q_2+c_4q_3+c_3q_4&=-c_7\\
c_5q_1+c_4q_2+c_3q_3+c_2q_4&=-c_6\\
c_4q_1+c_3q_2+c_2q_3+c_1q_4&=-c_5\\
c_3q_1+c_2q_2+c_1q_3+c_0q_4&=-c_4
\end{aligned}\right\} \qquad (2.6)$$

So the matrix associated with this method is given as:

$$\begin{bmatrix} c_6 & c_5 & c_4 & c_3\\ c_5 & c_4 & c_3 & c_2\\ c_4 & c_3 & c_2 & c_1\\ c_3 & c_2 & c_1 & c_0\end{bmatrix}\begin{bmatrix} q_1\\ q_2\\ q_3\\ q_4\end{bmatrix}=\begin{bmatrix} -c_7\\ -c_6\\ -c_5\\ -c_4\end{bmatrix} \qquad (2.7)$$

At integration point, $x=x_{n+1}$, and so, (2.2) becomes

$$U(x)=\sum_{r=0}^{\infty}c_r x_{n+1}^r \qquad (2.8)$$

But by Taylor series expansion of y_{n+1}, we have

$$\sum_{r=0}^{\infty}c_r x_{n+1}^r=\sum_{r=0}^{\infty}\frac{h^r y_n^{(r)}}{r!} \qquad (2.9)$$

Which gives

$$c_r x_{n+1}^r=\frac{h^r y_n^{(r)}}{r!} \text{ for } r=0,1,2,3,\ldots \qquad (2.10)$$

From (2.9), we have

$$c_r=\frac{h^r y_n^{(r)}}{r!\,x_{n+1}^r} \qquad (2.11)$$

Hence

$$c_0=y_n;\ c_1=\frac{h^1\cdot y_n^{(1)}}{1!\cdot x_{n+1}^1};\ c_2=\frac{h^2\cdot y_n^{(2)}}{2!\cdot x_{n+1}^2};\ c_3=\frac{h^3\cdot y_n^{(3)}}{3!\cdot x_{n+1}^3};\ c_4=\frac{h^4\cdot y_n^{(4)}}{4!\cdot x_{n+1}^4};\ c_5=\frac{h^5\cdot y_n^{(5)}}{5!\cdot x_{n+1}^5};\ c_6=\frac{h^6\cdot y_n^{(6)}}{6!\cdot x_{n+1}^6};\ c_7=\frac{h^7\cdot y_n^{(7)}}{7!\cdot x_{n+1}^7};$$

Then (2.7) becomes:

$$\begin{bmatrix}
\frac{1}{6!}\frac{h^6}{x_{n+1}^6} & \frac{1}{5!}\frac{h^5}{x_{n+1}^5} & \frac{1}{4!}\frac{h^4}{x_{n+1}^4} & \frac{1}{3!}\frac{h^3}{x_{n+1}^3}\\
\frac{1}{5!}\frac{h^5}{x_{n+1}^5} & \frac{1}{4!}\frac{h^4}{x_{n+1}^4} & \frac{1}{3!}\frac{h^3}{x_{n+1}^3} & \frac{1}{2!}\frac{h^2}{x_{n+1}^2}\\
\frac{1}{4!}\frac{h^4}{x_{n+1}^4} & \frac{1}{3!}\frac{h^3}{x_{n+1}^3} & \frac{1}{2!}\frac{h^2}{x_{n+1}^2} & \frac{h}{1!x_{n+1}}\\
\frac{1}{3!}\frac{h^3}{x_{n+1}^3} & \frac{1}{2!}\frac{h^2}{x_{n+1}^2} & \frac{h}{1!x_{n+1}} & 1
\end{bmatrix}\begin{bmatrix} q_1\\ q_2\\ q_3\\ q_4\end{bmatrix}=\begin{bmatrix}
\frac{1}{7!}\frac{h^7}{x_{n+1}^7}\\
\frac{1}{6!}\frac{h^6}{x_{n+1}^6}\\
\frac{1}{5!}\frac{h^5}{x_{n+1}^5}\\
\frac{1}{4!}\frac{h^4}{x_{n+1}^4}
\end{bmatrix} \qquad (2.12)$$

However the use of matrix (2.12), will give us investigative advantage to obtain results easily. When

(2.12) is subjected to a test equation $y' = \lambda y$, it becomes:

$$\begin{bmatrix} \frac{1}{6!}\frac{\bar{h}^6}{x_{n+1}^6} & \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} \\ \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} \\ \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} & \frac{\bar{h}}{x_{n+1}} \\ \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} & \frac{\bar{h}}{x_{n+1}} & y_n \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{7!}\frac{\bar{h}^7}{x_{n+1}^7} \\ \frac{1}{6!}\frac{\bar{h}^6}{x_{n+1}^6} \\ \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} \\ \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} \end{bmatrix} \quad (2.13)$$

Where $\bar{h} = \lambda h$

By using Crammer's rule on (2.13), we get the values for q_1, q_2, q_3 and q_4 as follows:

Let

$$A^* = \begin{bmatrix} \frac{1}{6!}\frac{\bar{h}^6}{x_{n+1}^6} & \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} \\ \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} \\ \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} & \frac{\bar{h}}{x_{n+1}} \\ \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} & \frac{\bar{h}}{x_{n+1}} & y_n \end{bmatrix} \quad (2.14)$$

$$X_1 = \begin{bmatrix} \frac{1}{7!}\frac{\bar{h}^7}{x_{n+1}^7} & \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} \\ \frac{1}{6!}\frac{\bar{h}^6}{x_{n+1}^6} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} \\ \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} & \frac{\bar{h}}{x_{n+1}} \\ \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} & \frac{\bar{h}}{x_{n+1}} & y_n \end{bmatrix} \quad (2.15)$$

$$X_2 = \begin{bmatrix} \frac{1}{6!}\frac{\bar{h}^6}{x_{n+1}^6} & \frac{1}{7!}\frac{\bar{h}^7}{x_{n+1}^7} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} \\ \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{6!}\frac{\bar{h}^6}{x_{n+1}^6} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} \\ \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} & \frac{\bar{h}}{x_{n+1}} \\ \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{\bar{h}}{x_{n+1}} & y_n \end{bmatrix} \quad (2.16)$$

$$X_3 = \begin{bmatrix} \frac{1}{6!}\frac{\bar{h}^6}{x_{n+1}^6} & \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{7!}\frac{\bar{h}^7}{x_{n+1}^7} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} \\ \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{6!}\frac{\bar{h}^6}{x_{n+1}^6} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} \\ \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{\bar{h}}{x_{n+1}} \\ \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & y_n \end{bmatrix} \quad (2.17)$$

$$X_4 = \begin{bmatrix} \frac{1}{6!}\frac{\bar{h}^6}{x_{n+1}^6} & \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} \\ \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{6!}\frac{\bar{h}^6}{x_{n+1}^6} \\ \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} & \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} & \frac{1}{5!}\frac{\bar{h}^5}{x_{n+1}^5} \\ \frac{1}{3!}\frac{\bar{h}^3}{x_{n+1}^3} & \frac{1}{2!}\frac{\bar{h}^2}{x_{n+1}^2} & \frac{\bar{h}}{x_{n+1}} & \frac{1}{4!}\frac{\bar{h}^4}{x_{n+1}^4} \end{bmatrix} \quad (2.18)$$

Therefore

$$q_1 = \frac{\det(X_1)}{\det(A^*)}, q_2 = \frac{\det(X_2)}{\det(A^*)}, q_3 = \frac{\det(X_3)}{\det(A^*)}, q_4 = \frac{\det(X_4)}{\det(A^*)}$$

With the use of Maple-18 package the following values were obtained:

$$q_1 x_{n+1} = -\frac{4h}{7}, q_2 x_{n+1}^2 = \frac{h^2}{7}, q_3 x_{n+1}^3 = -\frac{2h^3}{205}, q_4 x_{n+1}^4 = \frac{h^4}{1840}$$

From (2.1), when expanded, we have:

$$\frac{y_{n+1}}{y_n} = \frac{\left((1+A+B+C)+\bar{h}(1+A+B)+\frac{\bar{h}^2}{2}(1+A)+\frac{\bar{h}^3}{6}\right)}{1+A+B+C+D} \quad (2.19)$$

Where $A = q_1 x_{n+1}, B = q_2 x_{n+1}^2, C = q_3 x_{n+1}^3, D = q_4 x_{n+1}^4$

Substituting into (2.13) and simplifying, we get:

$$S(\bar{h}) = \frac{4\bar{h}^3 + 60\bar{h}^2 + 360\bar{h} + 840}{\bar{h}^4 - 16\bar{h}^3 + 120\bar{h}^2 - 480\bar{h} + 840} \quad (2.20)$$

And (2.14) is called the stability function of the method.

$$S(\bar{h}) = \frac{4\bar{h}^3 + 60\bar{h}^2 + 360\bar{h} + 840}{\bar{h}^4 - 16\bar{h}^3 + 120\bar{h}^2 - 480\bar{h} + 840} = \frac{\rho(\bar{h})}{\psi(\bar{h})} \quad (2.21)$$

By letting $\bar{h} = u + iv$

$$\rho(u,v) = 4(u+iv)^3 + 60(u+iv)^2 + 360u + 360iv + 840 \quad (2.22)$$

Expanding (2.16) and converting to the form $(A+iB)$, we obtain

$$A = 4u^3 - 12uv^2 + 60u^2 - 60v^2 + 360u + 840 \quad (2.23)$$

$$B = \left(12u^2 v - 4v^3 + 120uv + 360v\right) \qquad (2.24)$$

Similarly

$$\psi(u,v) = (u+iv)^4 - 16(u+iv)^3 + 120(u+iv)^2 - 480u - 480iv + 840 \qquad (2.25)$$

Expanding (2.25) and converting to the form $(A+iB)$, we obtain

$$C = u^4 - 6u^2v^2 + v^4 - 16u^3 + 48uv^2 + 120u^2 - 120v^2 - 480u + 840 \qquad (2.26)$$

$$D = \left(4u^3 v - 4uv^3 - 48u^2 v + 16v^3 + 240uv - 480v\right) \qquad (2.27)$$

From (2.21)

$S(\bar{h}) = \dfrac{\rho(\bar{h})}{\psi(\bar{h})} < 1$ holds if and only if

$\left|\rho(\bar{h})\right| < \left|\psi(\bar{h})\right|$ and

$$\left|\rho(\bar{h})\right| - \left|\psi(\bar{h})\right| < 0$$

$$\left|\rho(\bar{h})\right| = (A^2 + B^2)$$

$$\left|\psi(\bar{h})\right| = (C^2 + D^2)$$

$$\therefore\ (A^2 + B^2) - (C^2 + D^2) =$$

$$\begin{aligned}&\left(4u^3 - 12uv^2 + 60u^2 - 60v^2 + 360u + 840\right)^2 + \left(12u^2 v - 4v^3\right.\\ &\quad \left.+ 120uv + 360v\right)^2 - \left(u^4 - 6u^2v^2 + v^4 - 16u^3 + 48uv^2 + 120u^2\right.\\ &\quad \left.- 120v^2 - 480u + 840\right)^2 - \left(4u^3 v - 4uv^3 - 48u^2 v + 16v^3\right.\\ &\quad \left.+ 240uv - 480v\right)^2\end{aligned} \qquad (2.28)$$

Expanding we get:

$$\begin{aligned}&-u^8 - 4u^6v^2 - 6u^4v^4 - 4u^2v^6 - v^8 + 32u^7 + 96u^5v^2 + 96u^3v^4\\ &\quad + 32uv^6 - 480u^6 - 960u^4v^2 - 480u^2v^4 + 5280u^5 + 6720u^3v^2\\ &\quad + 1440uv^4 - 24960u^4 - 11520u^2v^2 + 192000u^3 + 57600uv^2\\ &\quad - 201600u^2 + 1411200u\end{aligned} \qquad (2.29)$$

Again converting (2.29) to polar form by letting $u = R\cos(\phi)$; $v = R\sin(\phi)$, we obtain:

$$\begin{aligned}F(R,\phi) = &-R^8\cos(\phi)^8 - 4R^8\cos(\phi)^6\sin(\phi)^2 - 6R^8\cos(\phi)^4\sin(\phi)^4\\ &- 4R^8\cos(\phi)^2\sin(\phi)^6 - R^8\sin(\phi)^8 + 32R^7\cos(\phi)^7\\ &+ 96R^7\cos(\phi)^5\sin(\phi)^2 + 96R^7\cos(\phi)^3\sin(\phi)^4\\ &+ 32R^7\cos(\phi)\sin(\phi)^6 - 480R^6\cos(\phi)^6 - 960R^6\cos(\phi)^4\sin(\phi)^2\\ &- 480R^6\cos(\phi)^2\sin(\phi)^4 + 5280R^5\cos(\phi)^5 + 6720R^5\cos(\phi)^3\sin(\phi)^2\\ &+ 1440R^5\cos(\phi)\sin(\phi)^4 - 24960R^4\cos(\phi)^4\\ &- 11520R^4\cos(\phi)^2\sin(\phi)^2 + 192000R^3\cos(\phi)^3\\ &+ 57600R^3\cos(\phi)\sin(\phi)^2 - 201600R^2\cos(\phi)^2 + 1411200R\cos(\phi)\end{aligned} \qquad (2.30)$$

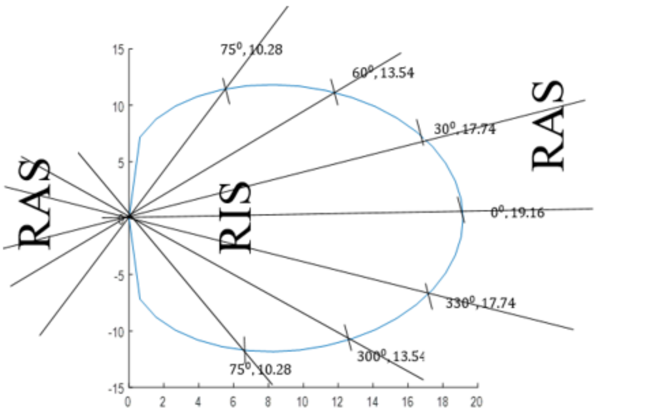
Next to determine the various values of R, for the corresponding values of ϕ for $0 \le \phi \le 2\pi$

We have the following table:

TABLE 1: Computed values of ϕ and R

ϕ	R	ϕ	R	ϕ	R	ϕ	R
0	19.16	5*pi/9	-8.91	10*pi/9	-18.53	5*pi/3	13.54
pi/36	19.12	7*pi/36	-10.28	41*pi/36	-18.17	61*pi/36	14.44
pi/18	19.00	11*pi/18	-11.48	7*pi/6	-17.74	31*pi/18	15.26
pi/12	18.80	23*pi/36	-12.56	43*pi/36	-17.24	7*pi/4	15.99
pi/9	18.53	2*pi/3	-13.54	11*pi/9	-16.65	16*pi/9	16.65
5*pi/36	18.17	25*pi/36	-14.44	5*pi/4	-15.99	65*pi/36	17.24
pi/6	17.74	13*pi/18	-15.26	23*pi/18	-15.26	11*pi/6	17.74
7*pi/36	17.24	3*pi/4	-15.99	47*pi/36	-14.44	67*pi/36	18.17
2*pi/9	**16.65**	7*pi/9	-16.65	4*pi/3	-13.54	17*pi/9	18.53
pi/4	15.99	29*pi/36	-17.24	49*pi/36	-12.56	23*pi/12	18.80
5*pi/18	15.26	5*pi/6	-17.74	25*pi/18	-11.48	35*pi/18	19.00
11*pi/36	14.44	31*pi/36	-18.17	17*pi/12	-10.28	71*pi/36	19.12
pi/3	13.54	8*pi/9	-18.53	13*pi/9	-8.91	2*pi	19.16
13*pi/36	12.56	11*pi/12	-18.80	53*pi/36	-7.21		
7*pi/18	11.48	17*pi/18	-19.00	3*pi/2	0.00		
5*pi/12	10.28	35*pi/36	-19.12	55*pi/36	7.21		
4*pi/9	8.91	p1	-19.16	14*pi/9	8.91		
17*pi/36	7.21	37*pi/36	-19.12	19*pi/12	10.28		
pi/2	0.00	19*pi/18	-19.00	29*pi/18	[illegible]		
19*pi/36	-7.21	13*pi/12	-18.80	59*pi/36			

When the values are plotted to get the Jordan curve, we have:



PLOT 1: Jordan plot of the method (k = 4)

RAS= Region of Absolute Stability

RIS- Region of Instability

From the above curve, the method for k = 4, is observed to be A-stable. It can further be shown that the method is also L-Stable, as can be seen below:

Proof:

Let

$$S(\bar{h}) = \frac{4\bar{h}^3 + 60\bar{h}^2 + 360\bar{h} + 840}{\bar{h}^4 - 16\bar{h}^3 + 120\bar{h}^2 - 480\bar{h} + 840} \quad (2.31)$$

Since

$$\lim_{\bar{h}\to\infty}\left(\frac{4h^3+60h^2+360h+840}{h^4-16h^3+120h^2-480h+840}\right) = 0 \quad (2.32)$$

Therefore, the method with the stability function (2.31) is L-Stable

III. Discussion and Conclusion

After a successful expansion of the rational integrator, we represented the Stability function in the form of polar curve using MATLAB and MAPLE-18 packages. Thus, the following findings are visible.

- It shows the region of absolute stability
- The exterior (outside) of the curve represent the region of absolute stability (RAS) of the integrator.
- The Region of instability (RIS) is on the positive side of the complex plane, which is within the stability (polar) curve.
- The rational integrator τ is within the interval ±19.16 where τ is the encroachment point.

It can be concluded that the Region of Absolute Stability (RAS) of the rational Interpolation method lies entirely on the Left-half of the complex plane. At the same time, the region of instability (RIS), lies within the Jordan curve as seen from the figure 1 above. This work revealed that, the Region of Absolute Stability (RAS) for k=4 is a superset of the entire left-half of the complex plane. Furthermore, the rational interpolation scheme is not only A-stable, but also L- stable.

References

[1] Aashikpelokhai U.S.U. and Elakhe, A.O. (2010). A model order numerical integrator for still differential system, Journal

[2] Aashikpelokhai U.S.U. (2000). A variable order numerical integrator based on rational interpolants. Journal of Nig. Mathematical Society. Vol. 19, 27- 28.

[3] Agbeboh, G.U. (2006). Comparison of some one-step integrator for solving singular initial value problems. Ph.D. Thesis, AAU.

[4] Agbeboh, G.U. and Aashikpelokhai U.S.U. (2002).An analysis of order, thirteen rational integrator. Journal of Sci. Engr. Tech Vol.9 (2), 4128- 4145.

[5] Fatunla S.O. and Aashikpelokhai, U.S.U (1994). A Fifth Order L-Stable Numerical Integrator. Scientific Comp pp. 68-86.

[6] Fatunla, S.O. (1982). Non-Linear One Step Methods for initial value problems. Comp. Method, Vol. 18, 231-239.

[7] Lambert, J.D. and Shaw, B (1965). On the Numerical Solution of y' = f(x, y). By a Clan of Formulae based on Rational Interpolant. Maths. Comp. Vol. 19. pp. 450 462.

[8] Lambert, J.D. (1973). Computational Methods in Ordinary Differential Equations, John Wiley and sons ltd. New York.

[9] Otunta, F.O and Ikhile, M.N.O (1999). Efficient Rational One-step Numerical Integrators for initial value problem in ODE. , J Comp. Method. Vol.72, pp. 49-61.

First Author: Ehiemua Michael Ebhodaghe is a lecturer at Ambrose Alli University, Ekpoma, Edo State. Nigeria. He is currently undergoing his Doctorate degree programme in the same institution in the area of Numerical Analysis, with specific interest in Runge-Kutta methods. Since the area of rational interpolation scheme are under Numerical Analysis, he has incorporated this study in his research area He has published articles in so many Journals in Nigeria and in the international journals.

Second Author: Agbeboh Goddy Ujagbe is a lecture at Ambrose Alli University, Ekpoma, Edo State. Nigeria. He has his Ph.D in the same University where he teaches in the area of Numerical Analysis with over 20 years as post lecturing experience. He has written books and articles in the area of Numerical Analysis with special interest in Runge-Kutta methods, locally, nationally and internationally.