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Construction of a Three-Dimensional Chess Board for Bishop Movement within the Forbidden Area with Vector Directives

M. Laisin, O. C. Okoli, E. I. Chukwuma, C. A. Okaa-Onwuogu.

Abstract- **In this paper, the authors reviewed that the game of chess is the most highly played game in most part of the world. However, focusing on studies about bishop moves on a chessboard with some current and relevant state of knowledge regarding the two-dimensional and three-dimensional movement of a bishop/rook on a board with forbidden space. We also stated some basic results on vectors that are helpful in analyzing and studying the movements of a bishop on a three-dimensional chessboard. In addition, we showed that the three-dimensional boards with l non-attacking bishop can generate a three-dimensional bishop polynomial with a generating function $\mathfrak{B}(x, B_1, B_2)$. Furthermore, some problems on bishop moves were solved by applying bishop generating functions.**

Index Terms—**Chess movements; Vectors; three-dimensional structures; Permutation;r-arrangement; combinatorial structures; Disjoined chess board in three-dimension; Vector generating function.**

I. INTRODUCTION

The game of chess is highly played in most part of the world and is considered as one of the most famous game that originated from Northwestern India in 6th century from an ancient Indian game called Chaturanga (Ashtapada). It is played on an 8×8 board with a total of 64 square. Even though, the Chaturanga game is not considered as a standard chess game compared to today's chess game that is played on a similar board with black and white squares having two players. However, in the 10th century chess game made a wonderful break through by spreading to Persia, and further extending to the Islamic Arabian Empire to Europe and to the Asian continent in places where colonization and civilization had moved to (Laisin, Chukwuma, Okeke, 2020). Although, the playing of chess game was highly discouraged during this period by rollers and Christian leaders, who considered the game as an educator of warring techniques. However, with complete license the game survived penalization from leaders of the Christian churches for an uncountable number of times during the end of the 15th Century (Laisin, Chukwuma, Okeke, 2020). Thus, in the year 1880, the chess game metamorphosed into different shape with completely different appearances which has developed to today's chess game. However, the period from 1880 – 1950 was considered as the romantic era for chess game with many national chess players round the globe. The players of this game depend on tactics, sacrifices techniques, technical and dynamic playing skills.

Furthermore, the game of chess is tagged as one of the most played World games with many fans but its origin is not easy to trace and it is completely controversial. Chess game has many kinds of legends, plain guesses, and stories together with a dispute over where the game originated from and when it started. Most people accepted that it was not an individual that discovered the establishment of the game of chess because the game is too complicated with all its concepts and rules for an individual mind to have discovered. Then, the chess game was in a steady flux until Steinitz Wilhelm was called as the first famous World official chess champion in 1886. However, a wise man from Indian under the roll of a famous tyrannical King called King Shahram (Laisin, Chukwuma, Okeke, 2020) was considered as one of the first man who discovered the game of chess in India. In addition, the wise man decided to explain to King Shahram how important the game is if everyone in his kingdom is having a good knowledge of how this game is played. Thus, the chess game he formulated and described to King Shahram includes the following; the King, his Queen, the knights, bishops, rooks and pawns that is played on an 8×8 board by two players. With this, the king was surprised to see that everyone in his kingdom was very important and well represented in the chess game. In-addition, King Shahram discovered that the game shows a warring kingdom where everyone is important and also playing their part to protect the King with his kingdom. The king instructed everyone in his kingdom to study and play the game (chess). King Shahram with gratitude thanked the wise man and also instructed and played the game with his generals, the game of chess started from there.

However, the first World official Championship for chess game occurred in the year 1886 in which Steinitz Wilhelm turns out to be the overall best official champion of the world. Since 1948, the world champion has been regulated by the federation international des echecs (FIDE), the games international governing body. Even though, there are varieties of hypothesis about the history of chess and there is no specific person who invented the popular game owning to its complexity and thus there are many interesting legends pertaining to its origin the game still stands as the most highly played game in the world. In-addition, during 20th century, the game of chess was revived with chess engine and database invention such that the game can further be played

M. Laisin, Department of Mathematics, Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria
O. C. Okoli, Department of Mathematics, Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria
C. A. Okaa-Onwuogu, Department of Mathematics, Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria
E. I. Chukwuma. Department of Mathematics, Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria

with computers. Thus, in 1997, Kasparov played and lost the World chess championship for a six times match to the IBM's computer Deep Blue. Thus, the website and online chess game was developed which later became so popular between the years 2007 – 2018 with many players round the world. In 2007, the website of chess game (chess.com) was introduced together with Lichess in 2010 and chess 24's website in 2014 (Laisin, Okeke, Chukwuma, 2020).

In addition, the game of chess is one of the oldest and most popular board games, that is played by two-players on a checkered board with 64 squares arranged in an 8×8 grid with alternating colors (usually white and black) as shown in fig1 below, with the current world champion as Magnus Carlsen of Norway.

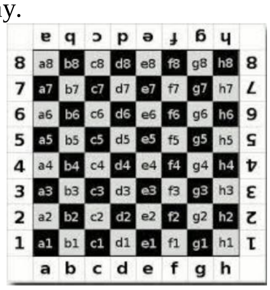

Fig. 1

Thus, each player begins with six different types of pieces: 8 pawns, 2 rooks, 2 bishops, 2 knights, a queen and the most important one the king all in the same color (Laisin, Chukwuma, Okeke). The objective of the game is to checkmate the opponent's king by placing it under an inescapable threat of capture. Each piece moves differently with the most powerful one been the queen and the least powerful the pawn.

The Bishop is a tall slender piece with pointed tip that has a strange cut made into it and it sits next to the knight piece. It has a value which is less than that of a rook. The bishop can move as many unoccupied squares as possible diagonally as far as there is no piece obstructing its path. Bishops capture opposing pieces by landing on the square occupied by an opponent piece. A bishop potential is maximized by placing it on an open, long diagonal such that it will not be obstructed by friendly pawn or an opponent's piece. A quick development of the bishop can be achieved by a special move called fianchetto. How a bishop gets along with pawns determines if it is a good or bad bishop. If your bishop and most of your pawn are on the same color squares then it is a bad bishop because it has fewer squares available to it. Each player starts out with two bishop pieces, each one residing on its own color of square. In addition, a bishop moves diagonally and captures a piece if that piece rests on a square in the same diagonal (LAISIN, 2018; Laisin, and Uwandu, 2019).

However, the polynomial for nonattacking bishop has a very good part to play in the theory of permutations with forbidden positions (Laisin, Okoli, &Okaa-onwuogu, 2019; Laisin&Uwandu, 2019; Laisin, 2018; LAISIN, 2018; Skoch, 2015; Jay &Haglund, 2000; Herckman, 2006; Chung, & Graham,1995) have shown that polynomial of either the bishop/rook on a given board can be generated recursively by applying cell decomposition techniques of Riordan (Abigail, 2004; Riordan, 1980; Riordan, 1958). Furthermore, Laisin, Okoli, &Okaa-onwuogu, 2019; Laisin&Uwandu, 2019; LAISIN, 2018; Laisin&Ndubuisi, 2017; Jay Goldman, and James Haglund, 2000 studied, examined and investigated movement of bishop/rook on a board with forbidden area to develop techniques for polynomials using generating functions. Thus, to determine the solutions for fundamental problems by examining the existence, enumeration and structure of the bishop/rook generating function on an $n \times m$ board. Informally, the bishop moves for a nonattacking bishop can be classified into three categories: search, generation, and enumeration (LAISIN, 2018; Bona, 2007).

The polynomials generated by nonattacking bishop provide a way of enumeration for permutation with forbidden positions that was developed by Kaplansky, and Riordan, 1946. LAISIN, 2018; Nickolas and Feryal, 2009 generalized these properties and theorems for two-dimensional bishop polynomials. However, more advanced dimensions were partially done for the three-dimensional cases (Laisin, Okeke, Chukwuma, 2020; Laisin, Chukwuma, Okeke, 2020; Zindle, 2007).

Now, we shall be focusing on the three-dimensional boards for non-attacking bishop to generate a three-dimensional bishop polynomial within the forbidden area (Michaels, 2013; Shanaz, 1999). In addition, we shall apply the bishop generating functions for a two– dimensional and the three-dimensional cases on disjoined sub-boards.

A. Basic definitions

A ring R is a set with two laws of composition $+$and $\times$ called addition and multiplication, which satisfy these axions;

a. With the composition $+, R$ is an abelian group, with identity denoted by 0. This abelian group is denoted by R^{+}
b. Multiplication is associative and has an identity denoted by 1.
c. Distributive law for all $a, b, c \in R, \Rightarrow (a+b)c = ac + bc \;\&\; c(a+b) = ca + cb$(Artin, 1991)

A chess board B of a ring is a chess board which is closed under the operations of addition subtraction, and multiplication and which contains the first placement ($b_0(B) = 1$). A bishop polynomial with forbidden positions is denoted as $\mathfrak{B}(x, B)$, given by

$$\mathfrak{B}(x, B) = \sum_{i=1}^{k} b_i(B)x^i,$$

where $\mathfrak{B}(x, B)$ has coefficients $b_i(B)$ representing the number of ways of bishop's placements on B. Furthermore, on m $\times n$ board B, we have $b_0(B) = 1$ and the coefficients are determined by

$$\mathfrak{B}(x, B) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k} k!\, x^k$$
$$= \sum_{k=0}^{\min(m,n)} \frac{n!\, m!}{k!\,(n-k)!\,(m-k)!} x^k.$$

(LAISIN, 2018)

B. Definition

Suppose that B be is an $m \times m$ board and its diagonal denoted by $\mathfrak{D}^\theta$ and let;

$$F(y_1, y_2, \dots, y_k) = \sum f(m_1, m_2, \dots, m_k) y_1^{m_1} y_2^{m_2} \dots y_k^{m_k} \in K[[(y_1, y_2, \dots, y_m)]]$$

Then, the $\mathfrak{D}^\theta$ is the power series in a single variable y defined by

$$\mathfrak{D}^\theta = \mathfrak{D}^\theta(y) = \sum_m f(m, m, \dots, m) y^m \qquad (\text{LAISIN}, 2018)$$

C. Standard Basis

Suppose $\mathfrak{D}^\theta = F^m$ is the space of diagonal vectors and let the diagonal vector be denote e_i with $b_0(B) = 1$ in the i^{th} position and zeros elsewhere. Then, the m vectors e_i from a basis for F^m . That is every vector $X = (x_1, x_2, \dots, x_k)$ has the unique expression;

$$XE = x_1 e_1 + x_2 e_2 + \dots + x_m e_m$$

as the linear combination of $E = (e_1, e_2, \dots, e_m)$ (LAISIN, 2018).

D. Vector quantities

These are those quantities that have both magnitude and direction.

E. Resultant vector

The resultant vector is that single vector which would have the same effect in magnitude and direction as the original vectors acting together.

F. Parallelogram law

The parallelogram law of vectors states that if two vectors are represented in magnitude and direction by the adjacent sides of a parallelogram drawn from the point of intersection of the vectors represents the resultant vector in magnitude and direction.

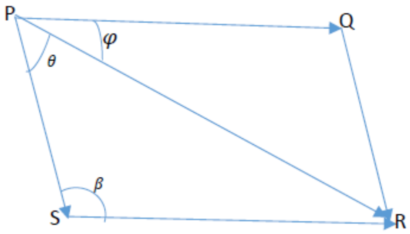

Fig 2

By cosine rule we have;

$$\boldsymbol{PR} = [(PQ)^2 + (QR)^2 - 2(PQ)(QR)cos\beta^\circ]^{\frac{1}{2}}$$

$$= [(PQ)^2 + (QR)^2 + 2(PQ)(QR)cos(180^\circ - \theta^\circ)]^{\frac{1}{2}}$$

By sine rule we have;

$$\frac{sin\theta^\circ}{SR} = \frac{sin\varphi^\circ}{PS} = \frac{sin\beta^\circ}{PR}$$

$$\boldsymbol{PR} = \frac{PSsin\beta^\circ}{sin\varphi^\circ} = \frac{SRsin\beta^\circ}{sin\theta^\circ}$$

II. THEOREM

If the movement on fig. 2.1 is a rook movement, then the angle between the vertical and the horizontal rook movement must be $90° - [\alpha, \beta, \theta]$.

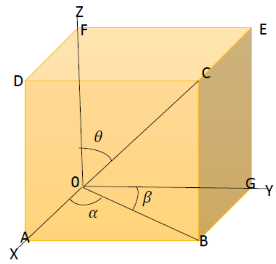

Fig. 2.1 (Laisin, Okeke, Chukwuma; 2020).

A. Theorem

Suppose M is the rook movement and the distance from a fixed point $F(v_1, v_2, v_3)$ to any point $T(s_1, s_2, s_3)$. Then, the ∇M is a unit vector in the direction of the rook movement $\boldsymbol{FT} = \boldsymbol{M}$ (Laisin, Okeke, Chukwuma; 2020).

B. Theorem: Angle between two vectors

Suppose γ is the angle between two vectors. Then, the sum of products of the corresponding direction of the rook movement is the cosines from the two generated vectors by the restricted area. Hence, the rook movement is a three-dimensional structure (Laisin, Chukwuma, Okeke; 2020).

C. Theorem

The number of ways to arrange n bishops among m positions $(m \geq n)$through an angle of $\theta = 45^0$for movement on the board with forbidden positions is;

$$\mathfrak{B}(y, B)P_{(m,\ n)} = \sum_{k=0}^{n} (-1)^k b_k^\theta\ P_{(m-k,\ n-k)}$$

Proof

The proof of theorem 3.1 follows immediately from Lemma 2.1 in arranging n bishops among m positions $(m \geq n)$through a direction of movement in an angle of 45^0 with forbidden positions is as follows;

Case 1$m > n$

$$\mathfrak{B}(y, B)P_{(m,\ n)} = P_{(m,\ n)} - b_1^\theta(B)P_{(m-1,n-1)} + b_2^\theta(B)P_{(m-2,n-2)} - b_3^\theta(B)P_{(m-3,n-3)} + \dots (-1)^m b_m^\theta(B)P_{(m-n,0)}$$

$$= \sum_{k=0}^{m} (-1)^k b_k^\theta\ P_{(m-k,n-k)}$$

Case 2 $m = n$

$$\mathfrak{B}(y, B)P_{(n,\ n)} = P_{(n,\ n)} - b_1^\theta(B)P_{(n-1,n-1)} + b_2^\theta(B)P_{(n-2,n-2)} - b_3^\theta(B)P_{(n-3,n-3)} + \dots (-1)^n b_n^\theta(B)P_{(0,0)}$$

$$= \sum_{k=0}^{n} (-1)^k b_k^\theta(B)P_{(n-k,n-k)}\ (\text{LAISIN}, 2018;\ \text{Abigail}, 2004)$$ ■

D. Theorem(n-disjoint sub-boards with movements through an angle of 45^0)

Suppose, B isan $n \times n$board ofdarkenedsquares with bishops that move through a direction of an angle of $\theta = 45^0$then, $\mathfrak{B}(x, B)$ for thedisjointsub-boards is;

$$\mathfrak{B}(x,B)=\sum_{i=0}^{n}\prod_{k=0}^{n}\mathcal{X}_{B_{j,K}}(x)^i b_i^{\theta}(B_j),\ j=1,2,\dots n \quad (\text{LAISIN}, 2018)$$

III. RESULTS

THEOREM 3.1

Suppose B is a three-dimensional disjoined bishop boards with forbidden squares and let l non-attacking bishop movements generate a bishop function, then, the generating function is;

$$\mathfrak{B}(x,B_1,B_2)=\mathfrak{B}(x,B_1)x^i+\mathfrak{B}(x,B_{j+1})\varphi sign(\boldsymbol{v_1}\times\boldsymbol{v_2}\cdot\boldsymbol{v_3})x^i,\ \forall j=1,2,\cdots,l-1$$

where the number of disjoint boards is denoted as φ.

Proof

Let B_1 be a two-dimensional $n\times n$ board, with $l$non-attackingbishop, then, we have the following;

$$\mathfrak{B}(x,B_1)=\sum_{i=0}^{l-1}(x)^i b_i^{\theta}(B_1)=1+xb_i^{\theta}(B_1)+x^2b_i^{\theta}(B_1)+\ldots+x^{t-1}b_{l-1}^{\theta}(B_1)$$

Now, considering B_1 as an 8×8 two-dimensional chessboard, then, the maximum number of bishops moves on the forbidden squares for non-attacking bishops is as follows;

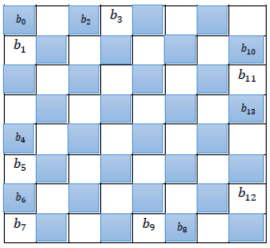

fig. 3.1

Thus, we have the following bishop placements for a two-dimensional chessboard;

$b_0(x,B_1)$, $b_1(x,B_1)$, $b_2(x,B_1)$, $b_3(x,B_1)$, $b_4(x,B_1)$, $b_5(x,B_1)$, $b_6(x,B_1),\cdots,b_{13}(x,B_1)$, then we have;

$$\sum_{i=0}^{13}(x)^i b_i^{\theta}(B_1)=1+xb_i^{\theta}(B_1)+x^2b_i^{\theta}(B_1)+\ldots+x^{t-1}b_{13}^{\theta}(B_1)$$
$$=b_0(x,B_1)+b_1(x,B_1)x+b_2(x,B_1)x^2+b_3(x,B_1)x^3+b_4(x,B_1)x^4+b_5(x,B_1)x^5+b_6(x,B_1)x^6+b_7(x,B_1)x^7+b_8(x,B_1)x^8+b_9(x,B_1)x^9+b_{10}(x,B_1)x^{10}+b_{11}(x,B_1)x^{11}+b_{12}(x,B_1)x^{12}+b_{13}(x,B_1)x^{13}=\mathfrak{B}(x,B_1)$$

Thus, this is the maximum number of bishop placements on an 8×8 two-dimensional chessboard. Similarly, as the board increases in size the total number of nonattacking bishops will also increase as new diagonals are introduced. Then, the following bishop placements can now follow; $b_0(x,B_1)b_1(x,B_1)$, $b_2(x,B_1)$, $b_3(x,B_1)$, $b_4(x,B_1)$, $b_5(x,B_1)$, $b_6(x,B_1)$, $\cdots$, $b_{l-1}(x,B_1)$ respectively.

However, the two-dimensional bishop boards generate the bishop function with the generating function.

$$\mathfrak{B}(x,B_1)=\sum_{i=0}^{l-1}[b_0(x,B_1),\ b_1(x,B_1),\ b_2(x,B_1),\ \cdots,\ b_{l-1}(x,B_1)]x^i$$
$$=b_0(x,B_1)+\ b_1(x,B_1)x+b_2(x,B_1)x^2+\ \cdots+\ b_{13}(x,B_1)x^{l-1}$$

$$\mathfrak{B}(x,B_1)=\sum_{i=0}^{13}(1,14,78,220,330,126,28,1,15,10,1,3,2,1)x^i$$

Thus,$\mathfrak{B}(x,B_1)$ is the maximum bishop generating function for an $n\times n$ two-dimensional board. Hence, a projection of the vector of $(l\boldsymbol{i}+l\boldsymbol{j}+l\boldsymbol{k}$on the vector $\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}$ gives a unit vector $\frac{1}{\sqrt{3}}(\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k})$ with a bishop projection movement as $\frac{3l}{\sqrt{3}}\ units$.

Now, considering the bishop movement on a 3-dimensional space, denoted as $\mathfrak{B}(x,B_1,B_2)$. We can now construct a 3-dimensional chess board that decomposes into disjoint boards with vectors $\boldsymbol{v_1}$,$\boldsymbol{v_2}$and$\boldsymbol{v_3}$ respectively.

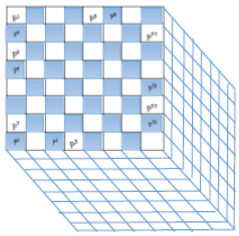

fig. 3.2

then, it follows the placement of bishops on the chess board, thus we have;

$$\mathfrak{B}(x,B_1,B_2)=\mathfrak{B}(x,B_1)x^i+\mathfrak{B}(x,B_{j+1})\varphi sign(\boldsymbol{v_1}\times\boldsymbol{v_2}\cdot\boldsymbol{v_3})x^i,\ \forall j=1,2,\cdots,l-1$$
$$=[\,[b_0(B_1)+\ b_1(B_1)+\cdots+b_{l-1}(B_1)]x^i+\ \varphi[b_0(B_2)+\ b_1(B_2)+b_2(B_2)+\cdots+b_{l-1}(B_2)]x^i$$

$$=\sum_{i=0}^{l-1}(x)^i b_i^{\theta}(\boldsymbol{x},B_1,B_2)$$

$$=\sum_{i=0}^{l-1}\prod_{S=0}^{l-1}b_i^{\theta}(B_j)\,\mathcal{X}_{B_j,S}x^i+\sum_{i=0}^{l-1}\prod_{S=0}^{l-1}\mathcal{X}_{B_j,S}b_i^{\theta}(B_{j+1})\varphi sign(\boldsymbol{v_1}\times\boldsymbol{v_2}\cdot\boldsymbol{v_3})x^i$$
$$=\sum_{i=0}^{l-1}\prod_{s=0}^{l-1}\mathcal{X}_{B_j,S}b_i^{\theta}(B_j)\,[1+\varphi sign(\boldsymbol{v_1}\times\boldsymbol{v_2}\cdot\boldsymbol{v_3})]x^i,\ \forall j=1,2,\cdots,l-1;$$

where the number of disjoint boards is denoted as φ.

■

IV. NUMERICAL APPLICATIONS

Example 4.1

A rice farm of eight square kilometers is to be worked by a maximum number of controllers and each controller works diagonally. If each diagonal plot can only be worked by one controller. How many controllers can be given this assignment and how in many ways?

Solution

Considering the rice farm of eight square kilometers, then, we have a maximum of 64 square plots with forbidden squares of non-attacking controllers is as follows;

b_0							
b_1							b_8
b_2							b_9
b_3							b_{10}
b_4							b_{11}
b_5							b_{12}
b_6							b_{13}
b_7							

Rice farm 4.1

Thus, we have 14 controller placements for plots as on the rice farm 4.1;

$b_0(x,B_1)$, $b_1(x,B_1)$, $b_2(x,B_1)$, $b_3(x,B_1)$, $b_4(x,B_1)$, $b_5(x,B_1)$, $b_6(x,B_1),\cdots,b_{13}(x,B_1)$, where the controller is denoted by $b_i: i = 0,1,2,3,\cdots,13$ then we have;

$$\mathfrak{B}(x,B_1) = b_0(x,B_1) + b_1(x,B_1)x + b_2(x,B_1)x^2 + \cdots + b_{13}(x,B_1)x^{l-1}$$

$$= \sum_{i=0}^{13}(1,14,78,220,330,126,28,1,15,10,1,3,2,1)x^i$$

$$= 1 + 14x + 78x^2 + 220x^3 + 330x^4 + 126x^5 + 28x^6 + x^7 + 15x^8 + 10x^9 + x^{10} + 3x^{11} + 2x^{12} + x^{13}$$

$$= 1 + 14P_{(13,13)} + 78(P_{(12,12)})^2 + 220(P_{(11,11)})^3 + 330(P_{(10,10)})^4 + 126(P_{(9,9)})^5 + 28(P_{(8,8)})^6 + (P_{(7,7)})^7 + 15(P_{(6,6)})^8 + 10(P_{(5,5)})^9 + (P_{(4,4)})^{10} + 3(P_{(3,3)})^{11} + 2(P_{(2,2)})^{12} + (P_{(1,1)})^{13}$$

$$= 65{,}529{,}875 \ ways$$

Thus, the rice farm can be given to 14 controllers and in 65,529,875 ways.

V. CONCLUSION

The polynomials generated by bishop movements on a forbidden space are very interesting for both two and three-dimensional cases. We were able to realize the objectives of this paper by showing that, the generating function for l non-attacking bishop movements generate a bishop function. Finally, we applied this formula to solve the rice farm problem.

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