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On Convergence Fields of Some Regular Matrix Transformations

Zakawat U. Siddiqui, Dauda I. Nkuno

***Abstract*—Some properties of convergence field of regular matrix transformation have already been established.The study of σ-porous sets has also been carried out. In this paper, we establish the conditions under which the convergence field F(A) of a regular transformation A is porous in S(A), the set of A-limitable real sequence.We also determine the conditions under which F(A) dense in S(A). Further, we prove that the convergence field ofvarious matrix transformation are close inS(A).**

Mathematics Classification: **40D09, 46B45**

***Index Terms*—Convergence field, Dense sets,Porosity, Regular and Irregular Matrix Transformations.**

I. Introduction

The concept of porosity in ℝ was used by Denjoy [1] in 1920under a different nomenclature.He proved that if P is a perfect nowhere dense subset of ℝ, then the set of points in P at which P is not *strongly bilaterally porous* is a first category subset of P. Dolženko [2] in 1967 introduced the notion of σ-porous sets and applied the notion to cluster sets of functions while in the year 1976, Šalát [3] showed the usefulness of the theorem of the discontinuity points of functions of the first Baire category in the study of the convergence field of a regular matrix method. Renfro [4] proved two strong versions of Denjoy statement for an arbitrary metric space. Kostyrko [5]in 2004 discussed some properties of the convergence field of regular matrix transformation of bounded sequences of real numbers. Visnyai [6] in 2006 proved a generalization of Steinhaus theorem for sequences of a Banach space and showed that the result of Šalàt [3] and Kostyrko [5] can be generalized for a space of sequences of element of a Banach space$(X, \|.\|)$. Later Kostyrko [7] in 2008 showed that the convergence field $F(A)$ of a regular matrix transformation is a σ-porous set in the metric space $S(A)$ endowed with Fréchet topology. Peter Letavaj [8] in 2012 showed that the convergence field $F(A)$ of a regular matrix transformation is a very porous set in S.

II. METHOD AND MATERIAL

In this paper we shall make use of the following material and known results.

Definition 2.1[9]: Suppose (X, ρ) is a metric space, $T \subset X$. Let $x \in X, \delta > 0$ and

$$B(x, \delta) = \{y \in X: \rho(x, y) < \delta\}$$

Zakawat U. Siddiqui, Department of Mathematical Sciences, University of Maiduguri, Maiduguri, Nigeria
Dauda I. Nkuno, Department of Mathematical Sciences, University of Maiduguri, Maiduguri, Nigeria.

Then, we put

$$\lambda(x, \delta, T) = sup\{t > 0: \exists z \in B(x, \delta) \; such \; that \; B(z, t) \subset B(x, \delta) \; and \; B(z, t) \cap T = \emptyset \}\}$$

If such $t > 0$ does not exist, we put $\lambda(x, \delta, T) = 0$.

The numbers

$$\underline{p}(x, T) = \lim_{\delta \to 0} inf \frac{\lambda(x, \delta, T)}{\delta}$$

and

$$\bar{p}(x, T) = \lim_{\delta \to 0} sup \frac{\lambda(x, \delta, T)}{\delta}$$

are called lower and upper porosity of the set T at x. If $\bar{p}(x, T) > 0$, then T is porous at x and if, for each $x \in X$, $\bar{p}(x, T) > 0$ *holds*, then T is porous in X. If $\underline{p}(x, T) > 0$, then T is very porous at $x$and if for each$x \in X$, $\underline{p}(x, T) > 0$ *holds*, then T is very porous in X.

Note that when a (divergent) sequence, say $\{x_n\}$ is acted upon by an infinite matrix $A = (a_{nk})$ say, the resultant sequence $\{t_n\}$, given by

$$t_n = \sum_{k=1}^{\infty} a_{nk} x_k \quad (1.1)$$

may exist. The set of all such sequences obtained for which the series in (1.1) converge is called the convergence field of the given matrix.

Therefore, if suppose S denotes the set of all the sequences of real or complex numbers and $S(A)$ is the set of all those sequences for which the sequence given by the series in (1.1) exists, then the convergence field of the matrix A is given by

$$F(A) = \{x \in S(A) | \; t \exists \lim_{n \to \infty} t_n = \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} x_k\} \quad (1.2)$$

It is well-known that the necessary and sufficient conditions under which the matrix method (A) is regular are given as:

i. $\sum_{k=1}^{\infty} |a_{nk}| < \infty$ for each $n = 1, 2, ...$
ii. $\lim_{n \to \infty} a_{nk} = 0$
iii. $\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = 1$

Theorem 2.1 [5]: Let (S, ρ) be the space of all bounded sequence of real or complex numbers endowed with the metric 'ρ'. Then, the convergence field F(A), is a very porous set in $S(A)$.

Theorem 2.2 [7]: Let A be a regular matrix, then the convergence field $F(A)$ of the matrix (A) is a σ-porous set in $S(A)$.

Theorem 2.3 [3]:Let $T = (a_{nk})$ and let (T) be a regular method. Let Y be a metric space and $Y_1(T)$ be the set of all T-limitable sequences of Y. Let M_k, k = 1, 2, ⋯ , be a non-void set of complex numbers. Let us suppose that

(j) $\sup_{k = 1,2 ...} diam \, M_k < +\infty$,

(ii) There exist two sequences
$y' = (\eta_k')_{k=1}^{\infty} \in Y_1(T)$ and $y" = (\eta_k")_{k=}^{\infty} \in Y_1(T)$
such that $\{\eta_k" - \eta_k'\}$is a convergence sequences andthat$\lambda = (\eta_k" - \eta_k') \neq 0$.
Then the set $Y_1(T)$ is a dense set (in Y) of the first Baire category.

III. MAIN RESULTS

Definition 3.1: Let$A = (a_{nk})$ be any infinite matrix and $x = (x_k)$ be any infinite sequence. Then, the convergence field $F(A)$ is close in $S(A)$ if $\forall\, x \in S(A) \backslash F(A)$ and $\forall\, \delta > 0, B(x,\delta) \cap F(A) = \emptyset$.

Theorem 3.1: Let $A = (a_{nk})$ be any infinite matrix and S be the space of all sequences of real or complex numbers. Then, $F(A)$ is porous in $S(A)$ if and only $A = (a_{nk})$ is irregular.

Proof. Suppose $F(A)$ is porous in $S(A)$. Then by definition 2.1, we have

$$P(x, F(A)) > 0$$

$$\Longrightarrow \lim_{\delta \to 0^+} sup \frac{\lambda(x,\delta,F(A))}{\delta} > 0$$

$$\Longrightarrow \lambda(x,\delta,F(A)) > 0$$

This implies that the set
$\{t > 0 : \exists y \in B(x,\delta) : B(y,t) \subset$
Bx,δ *and* $By,t\cap FA=\emptyset \neq \emptyset.$

Therefore, the number

$$sup\{t > 0; \exists y \in B(x,\delta); B(y,t) \subset B(x,\delta)\ and\ B(y,t) \cap F(A) = \emptyset\} > 0$$

$\Longrightarrow$ *there exists* $t > 0$ *such that* $B(y,t) \cap F(A) = \emptyset$

$\Longrightarrow B(y,t) \subset S(A) \setminus F(A)$

$\Longrightarrow$ that not all the series in 1.1 which exists converges.

$\Rightarrow A = (a_{nk})$is irregular

Conversely, suppose that $A = (a_{nk})$ is irregular. Then clearly, not all the series in 1.1 converge and therefore, $S(A) \setminus F(A) \neq \emptyset$. Thus for all $x \in S(A) \setminus F(A)$, there exist a number $\delta > 0$ such that $B(x,\delta) \subset S(A)$. Further, there exists $y \in B(x,\delta)$ such that for some numbers

$t_i > 0$, $B(y,t_i) \subset B(x,\delta)$ and $B(y,t_i) \cap F(A) = \emptyset$, $i = 1, 2, 3, \ldots$.

Therefore, there is a set

$$\{t > 0; \exists y \in B(x,\delta); B(y,t) \subset B(x,\delta)\ and\ B(y,t) \cap F(A) = \emptyset\} \neq \emptyset$$

Hence the number

$$sup\{t > 0; \exists y \in B(x,\delta); B(y,t) \subset B(x,\delta)\ and\ B(y,t) \cap F(A) = \emptyset\} > 0$$

$$\lambda(x,\delta,F(A)) = sup\{t > 0; \exists y \in B(x,\delta); B(y,t) \subset B(x,\delta)\ and\ B(y,t) \cap F(A) = \emptyset\} > 0$$

$$\Longrightarrow \lim_{\delta \to 0^+} sup \frac{\lambda(x,\delta,F(A))}{\delta} > 0$$

$$\Longrightarrow P(x,F(A)) = \lim_{\delta \to 0^+} sup \frac{\lambda(x,\delta,F(A))}{\delta} > 0$$

$$\Longrightarrow P(x,F(A)) > 0$$

Hence $F(A)$ is porous in $S(A)$ at all points $x \in S(A) \backslash F(A)$.

Theorem 3.2;If $A = (a_{nk})$ is any infinite matrix and S is the space of all sequences of real or complex numbers. Then, $F(A)$ is dense in $S(A)$ if and only if $F(A)$ is not porous in $S(A)$.

Proof.We prove this theorem by contradiction. Suppose that $F(A)$ is porous in $S(A)$, then, by definition 2.1, we have;

$$P(x,F(A)) > 0$$

$$\Longrightarrow \lim_{\delta \to 0^+} sup \frac{\gamma(x,\delta,F(A))}{\delta} > 0$$

$$\Longrightarrow \gamma(x,\delta,F(A)) > 0$$

This implies that the set

$$\{t > 0 : \exists y \in B(x,\delta) : B(y,t) \subset B(x,\delta)\ and\ B(y,t) \cap F(A) = \emptyset\} \neq \emptyset$$

And therefore the number

$$\Longrightarrow sup\{t > 0; \exists y \in B(x,\delta); B(y,t) \subset B(x,\delta)\ and\ B(y,t) \cap F(A) = \emptyset\} > 0$$

$\Longrightarrow$ *there exists* $t > 0$ *such that* $B(y,t) \cap F(A) = \emptyset$

$\Longrightarrow B(y,t) \subset S(A) \setminus F(A)$

$\Rightarrow A = (a_{nk})$ is irregular contradicting the fact that $A = (a_{nk})$ is regular.

Hence, $t > 0$ such that $B(y,t) \subset S(A) \setminus F(A)$ does not exist

$\Longrightarrow S(A) \setminus F(A) = \emptyset$

This implies that the set

$$\{t > 0 : \exists\ y \in B(x,\delta) : B(y,t) \subset B(x,\delta)\ and\ B(y,t) \cap F(A) = \emptyset\} = \emptyset$$

This implies that the number

$$\Longrightarrow sup\{t > 0 : \exists y \in B(x,\delta) : B(y,t) \subset B(x,\delta)\ and\ B(y,t) \cap F(A) = \emptyset\} = 0$$

$$\Longrightarrow \gamma(x,\delta,F(A)) = 0$$

$$\Longrightarrow \lim_{\delta \to 0^+} sup \frac{\gamma(x,\delta,F(A))}{\delta} = 0$$

$$P(x,F(A)) = 0$$

$\Rightarrow F(A)$is not porous in $S(A)$

$\Rightarrow$that for any point$y \in S(A)$ *and* $t >0$, $B(y,t) \cap F(A) \neq \emptyset$.

Hence, $F(A)$ is everywhere dense in $S(A)$ and $\overline{F(A)} = S(A)$.

Conversely, suppose $F(A)$is dense in $S(A)$. Then, for any $y \in S(A)$ there exist always a number $t > 0$ such that $B(y,t) \cap F(A) \neq \emptyset$

$\Longrightarrow S(A) \setminus F(A) = \emptyset$

$$\Rightarrow \nexists t > 0\ \forall\ x \in S(A)\ and\ \delta > 0\ \ such\ that\ \text{for any}\ y \in B(x,\delta)$$

$$B(y,t) \subset B(x,\delta)\ and\ B(y,t) \cap F(A) = \emptyset$$

Since $t > 0$ *does not exist, then,*

$$\Longrightarrow \gamma(x,\delta,F(A)) = 0$$

$$\Longrightarrow \lim_{\delta \to 0^+} sup \frac{\gamma(x,\delta,F(A))}{\delta} = 0$$

$$\Longrightarrow P(x,F(A)) = 0$$

$\Rightarrow F(A)$is not porous in $S(A)$and hence the proof.

Theorem 3.3: Let $A = (a_{nk})$ be any infinite matrix and S be the space of all sequences of real or complex numbers. Then, $F(A)$ is close in $S(A)$ if and only if $F(A)$ is porous in $S(A)$.

Proof:

Suppose that $F(A)$ is porous in $S(A)$. Then, Theorem 3.1, implies that $A = (a_{nk})$ is irregular and therefore,

$$P(x,F(A)) > 0$$

$$\overline{P}(x,F(A)) = \underline{P}(x,F(A))$$

$$\Longrightarrow \overline{P}(x,F(A)) = \lim_{\delta \to 0^+} sup \frac{\lambda(x,\delta,F(A))}{\delta} > 0$$

$\Rightarrow \lambda\big(x,\delta,F(A)\big) > 0$

This implies that the set

$$\{t > 0 : \exists y \in B(x,\delta) : B(y,t) \subset B(x,\delta)\ and\ B(y,t) \cap F(A) = \emptyset\} \neq \emptyset$$

$$\Rightarrow sup\{t > 0 ; \exists z \in B(x,\delta) ; B(z,t) \subset B(x,\delta)\ and\ B(z,t) \cap F(A)\} > 0$$

$$\Rightarrow there\ exist\ t > 0\ for\ some\ z \in B(x,\delta)\ such\ that\ B(z,t) \cap F(A) = \emptyset$$

$\Rightarrow S(A) \setminus F(A) \neq \emptyset$

$\Rightarrow \forall z \in S(A) \setminus F(A)\ \exists\ \varepsilon > 0$ z∈SA\FA ∃ ε>0 such that $B(z,\varepsilon) \subset S(A) \setminus F(A)$

$\Rightarrow S(A) \setminus F(A)$ is open. Hence $F(A)$ is closed in $S(A)$.

Conversely, supposed that $F(A)$ is closed in $S(A)$. Then, $S(A) \setminus F(A)$ is open in $S(A)$.

Hence, $\forall z \in S(A) \setminus F(A)$, there exist a number say $\varepsilon > 0$ such that $B(z,\varepsilon) \subset S(A) \setminus F(A)$. Thus, $\forall x \in S(A)$, there exist a number $\delta > 0$ such that $B(z,\varepsilon) \subset B(x,\delta)$,$B(z,\varepsilon) \cap F(A) = \emptyset$.

This implies that the set $\{\varepsilon > 0 ; \exists z \in B(x,\delta) ; B(z,\varepsilon) \subset$ *Bx,δ and Bz,ε⋂FA=∅≠∅.*

$\Rightarrow$

$sup\{\varepsilon > 0 ; \exists z \in B(x,\delta) ; B(z,\varepsilon) \subset$ *Bx,δ and Bz,ε⋂FA>0.*

$\Rightarrow \lambda\big(x,\delta,F(A)\big) > 0.$

$$P\big(x,F(A)\big) = \lim_{\delta \to 0^+} sup \frac{\lambda\big(x,\delta,F(A)\big)}{\delta}$$

$$= \lim_{\delta \to 0^+} inf \frac{\lambda\big(x,\delta,F(A)\big)}{\delta} >$$

$$\Rightarrow P\big(x,F(A)\big) > 0$$

$\Rightarrow F(A)$is porous in $S(A)$ at all the points $x \in S(A) \setminus F(A)$ and hence the theorem.

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