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Lie Symmetries, Exact Solutions and Hidden Symmetries of a Class of Kuramoto Sivashinsky(KS) Equations

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Abstract— **We study a class of non-linear partial differential equations, which describes the longitudinal motion of an elasto-plastic bar and anti-plane shearing deformation. In order to systematically explore the mathematical structure and underlying physics of the elasto-plastic flow in a medium, we generate all the geometric vector fields of the model equations. Using the classical Lie group method, it is shown that this equation does not admit space dilation type symmetries for a specific parameter value. The symmetry reductions and exact solutions to this equation are derived.**

Index Terms— **Lie Symmetries, invariants, and hidden symmetries**

I. INTRODUCTION

Partial differential equations (PDEs) (Ames, 1972) are widely used to describe complex phenomena in various fields of sciences, especially in physical sciences. Therefore, solving partial differential equation problems plays an important role in sciences. The Lie group method is one of the most powerful and fundamental methods used to determine invariant solutions of differential equations. It is applicable to both linear and nonlinear differential equations. The fundamental basis of this method is that when a differential equation is invariant under a Lie group of transformations (Olver, 1993), a reduction transformation exists. For PDEs with two independent variables, a single group reduction transforms the PDEs into ordinary differential equations (ODEs), which are generally easier to solve. Kuramoto-Sivashinsky equation introduced by Kuramoto in 1976, is one of the simplest physically, interesting spatially extended nonlinear systems that displays spatiotemporally chaotic behaviour. The equation is used to model stability of laminar flame fronts, phase turbulence in chemical oscillations and phase dynamics in reaction discusion systems (Kuramoto et. al., 1976), (Sivashinsky, Nicolaenko & Zaleski, 1977).

In this paper we are concerned with the following class of KS equations

$$u_{tt} + \alpha u_{xxxx} - \gamma(u_x^n)_x = 0, \qquad (1)$$

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where α,γ are constants and n > 0. This is a modified nonlinear wave equation introduced by Z. J. Yang and G. W. Chen, (Yang & Chen 2000). The equation (1) is associated with many equations. For example, in (Peire & An, 1995) a nonlinear wave equation

$$u_{tt} + \alpha u_{xxxx} - \gamma(u_x^2)_x = 0, \qquad (2)$$

where u(x,t) is the longitudinal displacement, α > 0, $\gamma \neq 0$ are real numbers was presented.

It is used to study some problems about vertical vibration of one dimensional elasticity pole and two dimensional anti-plane shear in the weak nonlinear analysis of micro-structure model in the elasticity and plasticity. Furthermore, the instability of its special solution and ordinary stain solution were studied in (Peire & An, 1995). In (Chen & Yang., 2000), (Zhang & Chen, 2003)., the authors considered the generalized equation of equation (2) and proved the existence and uniqueness of the global generalized solution and the global classical solution of several initial boundary value problems by the contraction mapping principle. The sufficient conditions of the nonexistence of the solution were also given. Z. Y. Yan (Yan, 2000). studied the equation (1) with the viscous damping term, by using the direct reduction method and obtained four new explicit solutions in the case of n = 2. The work of (Yan, 2000). was extended by (Wu, & Fan, 2007) via the same method and presented the solutions for the equation for $n \geq 3$. Since the nonlinear wave equation (1) has extensive physical applications in elasticity and plasticity, it is meaningful to study and investigate more new solutions of the equation.

In this paper, we have studied the equation via Lie symmetry analysis to generate more new solutions of the equation. Furthermore, the hidden symmetry approach is applied to generate new solutions that would not be predicted from the Lie point symmetries of the original wave equations.

The paper is structured as follows. In Section II, we present some basic operators and definitions which shall be used in our work. In Section III, the Lie point symmetries of equation (1) are presented. In Section IV, symmetry reductions and exact solutions are discussed. In Section V, the hidden symmetries are obtained. Discussions and concluding remarks are given in Section VI.

II. PRELIMINARIES

We present some basic notations, operators and definitions to be used in our work, which are taken from (bragimov, 1994)., (Kara & Mahomed., 1997), (Naz, Mahomed & Mason, 2008) (Okeke, Narain, & Govinder, 2019).

Definition 2.1.
A k^{th} − order (k ≥ 1) System E^{σ} of s partial differential equations of n independent variables x^i, $i = 1,2,\ldots,n$ and m-dependent variables u^{α} : $\alpha = 1,2,\ldots,m$ is defined by;

$$E^{\sigma}\left(x^i,u^{\alpha},u_{(1)},\cdots,u_{(k)}\right)=0,\quad \sigma=1,\cdots,s \qquad (3)$$

where $u_{(1)},\ldots,u_{(k)}$ denote the collection of all first, second, ..., kth-order partial derivatives.

Definition 2.2.
The Euler-Lagrangian operator is defined by

$$\frac{\delta}{\delta u^{\alpha}}=\frac{\partial}{\partial u^{\alpha}}+\sum_{s\geq 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^{\alpha}_{i_1\cdots i_s}},\alpha=1,2,\ldots,m \qquad (4)$$

where

$$D_i=\frac{\partial}{\partial x^i}+u^{\alpha}_i\frac{\partial}{\partial u^{\alpha}}+u^{\alpha}_{ij}\frac{\partial}{\partial u^{\alpha}_j}+\cdots,\quad i=1,2,\ldots,n \qquad (5)$$

is the total derivative operator with respect to x^i

Definition 2.3.
The Euler-Lagrangian equations, associated with (3) are the equations

$$\frac{\delta L}{\delta u^{\alpha}}=0,\ \alpha=1,2,\ldots,m \qquad (6)$$

where L is referred to as a Lagrangian of (3).

Definition 2.4.
A Lie Backlund operator X is defined by

$$X=\xi^i\frac{\partial}{\partial x^i}+\eta^{\alpha}\frac{\partial}{\partial u^{\alpha}}+\sum_{s\geq 1}\varsigma^{\alpha}_{i_1\cdots i_s}\frac{\partial}{\partial u^{\alpha}_{i_1\cdots i_s}},\quad \alpha=1,2,\ldots,m \qquad (7)$$

where $\varsigma^{\alpha}_{i_1\cdots i_s}$ are given as

$$\varsigma^{\alpha}_i=D_i(\eta^{\alpha})-\varsigma^{\alpha}_{i_1\cdots i_s}D_i\varepsilon^j,\varsigma^{\alpha}_{i_1\cdots i_s}=D_{i_s}(\varsigma^{\alpha}_{i_1\cdots i_s})-u^{\alpha}_{ji_1\cdots i_{s-1}}D_{i_s}(\varepsilon^j),s\geq 1. \qquad (8)$$

The Lie point symmetry of equation (3) is a generator X of the form (7) that satisfies

$$X^{[k]}F_{|F=0}=0, \qquad (9)$$

where $X^{[k]}$ is the kth prolongation of X i.e.,

$$X^{[k]}=\xi^i(x,u)\frac{\partial}{\partial x^i}+\eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}}+\varsigma^{\alpha}_i(x,u,u_{(1)})\frac{\partial}{\partial u^{\alpha}_i}+\cdots+\varsigma^{\alpha}_{i1\ldots ik}(x,u,u(k))\frac{\partial}{\partial u^{\alpha}_{i1\ldots ik}}. \qquad (10)$$

III. LIE POINT SYMMETRIES

In this section, we present the Lie point symmetry generators admitted by (1). The Lie point symmetries admitted by (1) are generated by a vector field of the form

$$X=\xi^1(t,x,u)\frac{\partial}{\partial t}+\xi^2(t,x,u)\frac{\partial}{\partial x}+\eta(t,x,u)\frac{\partial}{\partial u}. \qquad (11)$$

The operator X satisfies the Lie symmetry condition (Kara, & Mahomed, 1997)), (Okeke, Narain, & Govinder, 2018)

$$X^{[4]}[u_{tt}+\alpha u_{xxxx}-\gamma(u^n_x)_x]|_{(1)}=0, \qquad (12)$$

where $X^{[4]}$ is the fourth prolongation of the operator X and can be computed from (10).

Expansion and separation of (12) with respect to the powers of different derivatives of u yields an overdetermined system in the unknown coefficients ξ^1,ξ^2 and η. Solving the overdetermined system for $\xi^1(t,x,u),\xi^2(t,x,u)$ and $\eta(t,x,u)$ we obtain

$$\xi^1(t,x,u)=C_1+tC_3,\ \xi^2(t,x,u)=C_2+\frac{1}{2}xC_3,$$
$$\eta(t,x,u)=\frac{1}{2}\left(\frac{n-3}{n-1}\right)uC_3+C_4+tC_5, \qquad (13)$$

where C_1,C_2,C_3,C_4 and C_5 are arbitrary constants. Solutions (13) prompt the consideration of special cases of equation (1) namely, (i) n = 1 and (ii) n = 3. Hence, we shall also discuss the symmetries based on these two cases. Equations (13) lead to a five-dimensional Lie algebra spanned by the following basis

$$X_{11}=\frac{\partial}{\partial x},\ X_{12}=\frac{\partial}{\partial t},\ X_{13}=\frac{x}{2}\frac{\partial}{\partial x}+t\frac{\partial}{\partial t}+\frac{1}{2}\left(\frac{n-3}{n-1}\right)u\frac{\partial}{\partial u},$$
$$X_{14}=\frac{\partial}{\partial u},\ X_{15}=t\frac{\partial}{\partial u}, \qquad (14)$$

The commutation relation between these five operators is presented in Table 1, where each entry, A_{ij}, constitutes the Lie Bracket $[X_{1i},X_{1j}]$ of two infinitesimal generators from (14) for $1\leq i,j\leq 5$

Special cases

(i) n = 1

This case gives rise to the linear wave equation

$$u_{tt}+\alpha u_{xxxx}-\gamma u_{xx}=0. \qquad (15)$$

Equation (15) admits four Lie point symmetries given by

$$X_{21}=\frac{\partial}{\partial x},\quad X_{22}=\frac{\partial}{\partial t},\quad X_{23}=u\frac{\partial}{\partial u}, \qquad (16)$$

as well as an infinite symmetry $X_{24}=F_1(t,x)\frac{\partial}{\partial u}$, where $F_1(t,x)$ is a solution of equation (15) and hence called the solution symmetry. This symmetry always arises in the event that the equation in question is linear. We observe the loss of invariance under dilations in space and time (X_{13}). This loss is indeed well compensated by the infinite-dimensional Lie subalgebra (X_{24}).

(ii) n = 3

In this case we obtain a well known PDE, the modified Boussinesq equation given by (Ghil & Paldor, 1994)

$$u_{tt}+\alpha u_{xxxx}-\gamma(u^3_x)_x=0. \qquad (17)$$

which was presented in the famous Fermi-Pasta-Ulam problem. The symmetries are

$$X_{31}=\frac{\partial}{\partial t},X_{32}=\frac{\partial}{\partial x},X_{33}=\frac{x}{2}\frac{\partial}{\partial x}+t\frac{\partial}{\partial t},X_{34}=\frac{\partial}{\partial u},X_{35}=t\frac{\partial}{\partial u}. \qquad (18)$$

Equation (17) is used to investigate the behavior of systems which are primarily linear but a nonlinearity is introduced as a perturbation. It also arises in other physical applications. In (Qu, 1988), three types of symmetry reductions of equation (17) were derived and it was shown tha the equation is unintegrable. The soliton solutions of some special cases of equation (17) were obtained in (Berryman, 1976), (Bona, 1975), (Liu, 1993), (Makhankov, 1978). by various techniques.

Table 1: Commutator table of the Lie algebra of (1).

[,]	X_{11}	X_{12}	X_{13}	X_{14}	X_{15}
X_{11}	0	0	X_{11}	0	X_{14}
X_{12}	0	0	$\frac{1}{2}X_{12}$	0	0
X_{13}	$-X_{11}$	$-\frac{1}{2}X_{12}$	0	$-\frac{1}{2}\left(\frac{n-3}{n-1}\right)X_{14}$	$\frac{1}{2}\left(\frac{n+1}{n-1}\right)X_{15}$
X_{14}	0	0	$\frac{1}{2}\left(\frac{n-3}{n-1}\right)X_{14}$	0	0
X_{15}	$-X_{14}$	0	$-\frac{1}{2}\left(\frac{n+1}{n-1}\right)X_{15}$	0	0

IV. SYMMETRY REDUCTION AND EXACT/INVARIANT SOLUTION

One of the main objectives for calculating the symmetries of differential equations is to use them to reduce the differential equations which could be solved to obtain exact solutions.

In this section, we apply the symmetries calculated in Section 3 to obtain reduced wave equations and exact solutions where possible. In particular, we use the translation generators in t variable given as $X_{22} = \frac{\partial}{\partial t}$

Case (i) n = 1

Case (i) n = 1

The reduction using X_{22} reduces (15) to fourth order linear ordinary differential equation in x given as

$$\alpha u'''' - \gamma u'' = 0 \qquad (19)$$

whose general solution is

$u(x) = C_1 e^{-x\sqrt{\frac{\gamma}{\alpha}}} + C_2 e^{x\sqrt{\frac{\gamma}{\alpha}}} + xC_3$, C_1, C_2, C_3 and C_4 are constants.

Case (ii) n = 2

With the same generator X_{22}, (1) reduces to the following fourth order non linear ordinary differential equation in x variable

$$\alpha u'''' - 2\gamma u'' u' = 0. \qquad (20)$$

The solution of (20) is

$$u(x) = C_1 - 6^{\frac{2}{3}}\, WeierstrassZeta\left[\frac{\left(\frac{\gamma}{\alpha}\right)^{\frac{1}{3}}(x+C_2)}{6^{\frac{1}{3}}}, \left\{\frac{2\times 6^{\frac{1}{3}} C_3}{\left(\frac{\gamma}{\alpha}\right)^{\frac{1}{3}}}, C_4\right\}\right]$$

Case (iii) n = 3

Similar approach reduces (17) to

$$\alpha u'''' - 3\gamma u'' u'^2 = 0. \qquad (21)$$

The general solution of (21) is an elliptic function given as,

$$u(x) = \frac{2}{3}\sqrt{2}\left(\sqrt{\alpha}\left(\frac{\gamma}{C_1}\right)^{-\frac{1}{4}} EllipticF\left(N\left(\frac{\gamma}{C_1}\right)^{\frac{1}{4}}, I\right) + x + C_2\right)^{\frac{3}{2}} + C_3.$$

where C_1, C_2 and C_3 are constants.
constants.

V. THE HIDDEN SYMMETRIES

Since we are interested in improving the solutions of the wave equations, it may also be necessary to look into some hidden symmetries of the wave equations.

It has been found that the reduced equation can admit more Lie point symmetries than the ones of the original equation. These new Lie point symmetries have been termed Type II hidden symmetries. On the other hand, it is possible that the reduced equation loses a Lie point symmetry admitted by the original equation. This lost symmetry is called Type I hidden symmetry (Abraham-Shrauner & Govinder, 2006), (Abraham-Shrauner & Govinder, 2008).

Case (i) n = 1

The Lie point symmetries of the reduced equation (19) are

$H_{21} = \frac{\partial}{\partial x}$, $H_{22} = \frac{\partial}{\partial u}$, $H_{23} = u\frac{\partial}{\partial u}$, $H_{24} = x\frac{\partial}{\partial u}$, $H_{25} = e^{-\sqrt{\frac{\gamma}{\alpha}}}\, x\frac{\partial}{\partial u}$, $H_{26} = e^{\sqrt{\frac{\gamma}{\alpha}}}\, x\frac{\partial}{\partial u}$.

The symmetries of (19) H_{21}, H_{22}, H_{23} and H_{24} are inherited symmetries from X_{21}, X_{23} and X_{24} of (15) respectively. The remaining two symmetries H_{25} and H_{26} are new symmetries obtained from the reduction and are referred to as Type II hidden symmetries.

Case (ii) n = 3

From the reduced equation (20), we obtain the following Lie point symmetries

$H_{31} = \frac{\partial}{\partial x}$, $H_{32} = \frac{\partial}{\partial u}$, $H_{33} = u\frac{\partial}{\partial u} - x\frac{\partial}{\partial x}$.

The reduced equation (21) inherits all the symmetries H_{31}, H_{32} and H_{33} from X_{32}, X_{34} and X_{33} of (17) respectively. Therefore, there is no Type II hidden symmetry, instead we have Type I hidden symmetries and they are X_{31} and X_{35}

VI. DISCUSSION AND CONCLUSION

In this paper, a nonlinear PDE found in elasto-plastic ow is studied using the group methods.

The Lie point symmetries of the model equations were derived. We found that the analysed model does not admit space dilation type symmetries as a result of the linearity of the equation when n = 1. Using the Lie point symmetries, we gave some symmetry reductions and obtained the invariant solutions of the equations.

Furthermore, the hidden symmetry approach was applied to generate more new solutions that would not be predicted from the Lie point symmetries of the original wave equations.

These results can be used to study deformity in elastic and plastic medium.

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