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Equivalence of the Defining Real Sequences for Spaces of Ultra Distributions

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***Abstract*— In this paper, we prove that $M_p^* \simeq M_p$ and reprove that $M_p^* p! \sim M_p$.**

The consequent result is a stronger version of the conjecture $M_p \subset p!\, M_p^*$ which has been established by H. Komatsu [1].

***Index Terms*—About four key words or phrases in alphabetical order, separated by commas.**

§1.0 Preliminaries

Let M_p be a weight sequence of positive real numbers with the initial condition that

$M_0 = 1$ and $M_p > 0$ for all $p \in \mathbb{N}_0$ satisfying the given two conditions:

(M.1) $\frac{M_p}{M_{p+1}} \leq \frac{M_{p-1}}{M_p}, p = 1, 2, 3, \cdots, M_p, M_{p+1} \neq 0.$

(M.2) $\sum_{k=p+1}^{\infty} \frac{M_{k-1}}{M_k} \leq Hp \frac{M_p}{M_{p+1}}$ for some positive real number $p = 1, 2, 3, \cdots$

Remark 1

The condition (M.1) can also be established as $\sum_{k=p+1}^{\infty} \frac{M_{k-1}}{M_k} \leq Hp \frac{M_{p-1}}{M_p}$. Also, $p! \subset CL^p M_p$ for some or all real numbers C and L.

Definition 1. We say that two real sequences M_p and N_p are equivalent (denoted

by $M_p \sim N_p$) if there exist some real positive constants A, B such that the following

condition holds:

$$A^p N_p \leq M_p \leq B^p N_p, p \in \mathbb{N}$$

We can also say that the real sequence M_p is equivalent to the real sequence N_p if

there exist a constant A such that $N_p \simeq A^p M_p$ and vice-versa.

Remark 2: The sequence of real numbers defined above with the given two con-

ditions are usually used to define various spaces of ultradifferentiable functions and

spaces of ultradistributions. These spaces are invariant with respect to the equivalence

((∼ " (see Komatsu [1] for more details). The essence of this work is to show

that $M_p^* \simeq M_p$ and reprove $M_p^* p! \sim M_p$ which refines the Theorems 11.5 and 11.8

in Komatsu [1].

Here let

$$M_p = \sup \underline{p!\, t^p}$$

$t \in \mathbb{N} \exp(M^*(t))$'

$$M_p^* = \sup \frac{t^p}{\exp(M^*(t))}$$

Where $M^* t$, the associated function of M_p^*, given as $M^*(t) = \sup \log\left(\frac{p!\, t^p}{M_p}\right)$, and $M(t)$, associated function of M_p as $M(t) = \sup \log\left(\frac{t^p}{M_p}\right)$.

Theorem 1. If a sequence M_p satisfies the conditions (M.1) and (M.2). Then

$M_p^* \simeq M_p, p \in \mathbb{N}andM_p^* p! \sim M_p$.

Proof. We first prove that $M_p^* p! \sim M_p$.

sequence $l_p = \frac{p}{m_p} + \sum_{k \geq p}^{\infty} \frac{1}{m_k}$.

Here

$$l_p = \frac{p}{m_p} + \sum_{k \geq p}^{\infty} \frac{1}{m_k} \infty$$

$$= \frac{p}{\frac{M_p}{M_{p-1}}} + \sum_{k \geq p}^{\infty} \frac{1}{\frac{M_k}{M_{k-1}}}$$

$$= \frac{pM_{p-1}}{M_p} + \sum_{k \geq p}^{\infty} \frac{M_{k-1}}{M_k}$$

To do this, let $m_p = \frac{M_p}{M_{p-1}}$, and let the

$$= \frac{pM_{p-1}}{M_p} + \frac{M_{p-1}}{p} + \sum_{k=p+1}^{\infty} \frac{M_{k-1}}{M_k}$$

$$= \frac{M_{p-1}}{M_p}(p+1) + \sum_{k=p+1}^{\infty} \frac{M_{k-1}}{M_k} \quad (1.1)$$

Since M_p satisfies the conditions (M.1) and (M.2), we have

$$\frac{M_{p-1}}{M_p}(p+1) + \sum_{k=p+1}^{\infty} \frac{M_{k-1}}{M_k} \leq \frac{M_{p-1}}{M_p}(p+1) + Hp \frac{M_p}{M_{p+1}}$$

$$\geq l_{p+1}$$

This implies that $l_p \geq l_{p+1} \Rightarrow l_p$ is a decreasing (monotone) and automatically

bounded. Now since M_p satisfies the condition: there is a constant $A \geq 1$ such that

$\frac{m_p}{PA} \leq \frac{1}{l_p} \leq \frac{m_p}{p}$. Since $l_p \leq \frac{M_{p-1}}{M_p}$, this implies that $l_p \leq \frac{1}{m_p}$. This gives us

$\frac{1}{l_p} \geq m_p \Rightarrow \frac{m_p}{pA} \leq \frac{1}{l_p} \leq \frac{m_p}{p}$, for $p \in \mathbb{N}$ (1.2)

Having established the fact that $l_{p+1} \leq l_p$, let $n_p = \frac{pl_1}{l_p}$.

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From (1.2) , this implies that $\frac{1}{l_p} = \frac{m_p}{pl_1} \Rightarrow m_p l_p = pl_1$, then we got $n_p = \frac{pl_1}{l_p}$.

Let $N_p = n_0 \times n_1 \times n_2 \times \cdots \times n_p = \prod_{j\in\mathbb{N}_0}^{p} n_j = \prod_{j\in\mathbb{N}_0}^{p} \frac{jj\,l_1}{l_j} = l_1 \prod_{j\in\mathbb{N}_0}^{p} \frac{jj}{l_j}$ with $n_0 = 1$.

Therefore, N_p is equivalent to M_p and satisfies condition (M.1): $N_p^2 \leq N_{p+1}N_{p-1}$ where $N_{p+1} = l_1 \prod_{j\in\mathbb{N}_0}^{p+1} \frac{j}{l_j}$ and $N_{p-1} = l_1 \prod_{j\in\mathbb{N}_0}^{p-1} \frac{j}{l_j} \Rightarrow (l_1 \prod_{j\in\mathbb{N}_0}^{p} \frac{j}{l_j})^2 \leq$ $l_1 \prod_{j\in\mathbb{N}_0}^{p+1} \frac{j}{l_j} l_1 \prod_{j\in\mathbb{N}_0}^{p-1} \frac{j}{l_j} = l_1^2 \prod_{j\in\mathbb{N}_0}^{p+1} \frac{j}{l_j} \prod_{j\in\mathbb{N}_0}^{p-1} \frac{j}{l_j}$.

Then $(l_1 \prod_{j\in\mathbb{N}_0}^{p} \frac{j}{l_j})^2 \leq \prod_{j\in\mathbb{N}_0}^{p+1} \frac{j}{l_j} \prod_{j\in\mathbb{N}_0}^{p-1} \frac{j}{l_j}$ and also satisfies condition (M.2). Since N_p is equivalent to M_p, it implies also that $\frac{N_p}{p!}$ satisfies condition (M.1) (that is, $(\frac{N_p}{p!})^2 \leq \frac{N_{p-1}}{p!}\frac{N_{p+1}}{p!} \Rightarrow (N_p)^2 \leq (p!)^2 \frac{N_{p-1}N_{p+1}}{(p!)^2} \Rightarrow N_p^2 \leq N_{p-1}N_{p+1}$) . It then follows from the above facts and the Gorny's theorem (Mandelbrojt [2]) that

$\frac{N_p}{p!} = \sup \frac{t^p}{\exp\,(N^*(t))}$, where $N_p = \sup \frac{t^p}{\exp\,(N^*(t))}$.

Thus

$$\frac{N_p}{p!} = \frac{1}{p!}N_p$$
$$= \frac{1}{p!} \sup \frac{p!\,t^p}{\exp\,(N^*(t))}$$
$$= \sup \frac{p!\,t^p}{p!\,\exp\,(N^*(t))}$$
$$= \sup \frac{t^p}{\exp\,(N^*(t))}$$

where $N^*(t)$ is an associated function of the sequence N_p.

Conversely, the equivalence $N_p \sim M_p$ implies that $\frac{N_p}{p!} \sim \frac{M_p}{p!}$ and there are constants $A, B > 0$ such that

$N^*(At) \leq M^*(t) \leq N^*(Bt)$, $t \in \mathbb{N}$ (1.3)

which indicates that the associated functions of the sequences M_p and N_p are equivalent from $A^p N_p \leq M_p \leq B^p N_p$.

Therefore,

$$(\frac{1}{B})^p \frac{N_p}{p!} = \frac{1}{B^p}\frac{N_p}{p!}$$
$$= \frac{1}{B^p} \sup \frac{t^p}{\exp\,(N^*(t))} \leq \sup \frac{t^p}{\exp\,(N^*(Bt))}$$

$\leq \sup \frac{t^p}{\exp\,(M^*(t))}$ (By applying $\frac{1}{N^*(Bt)} \leq \frac{1}{M^*(t)} \leq \frac{1}{N^*(At)}Big)$

$$= M_p^* \leq \sup \frac{t^p}{\exp\,(N^*(At))} = \frac{1}{A^p}\frac{N_p}{p!}$$

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Thus $M_p^* \leq \frac{1}{A_p}\frac{N_p}{p!}, p = 1,\ 2,\ 3, \cdots$ But since $\frac{N_p}{p!} \sim \frac{M_p}{p!}$. It follows that $M_p^* \sim \frac{N_p}{p!} \sim \frac{M_p}{p!} \Rightarrow M_p^* \sim \frac{M_p}{p!}$. Therefore $p! \subset M_p$.

Next we show that $M_p^* \simeq M_p$.

Given the associated functions $M(t)$ and $M^*(t)$ of the real sequences M_p and M_p^* respectively as $M(t) =$suplog $(\frac{t^p}{M_p})$, for $t > 0$ and $M^*(t) =$ suplog $(\frac{t^p}{M_p^*})$. we want to establish M_p^* from $M^*(t) =$ suplog $(\frac{t^p}{M_p^*})$.

$\exp\,(M^*(t)) = \exp$ (suplog $(\frac{t^p}{M_p^*})$)

$\leq$supexp (log $\frac{t^p}{M_p^*}$) $=$ sup $\frac{t^p}{M_p^*}$, $\exp\,(M^*(t)) = \frac{1}{M_p^*}$ sup t^p

Thus $M_p^* = \sup \frac{t^p}{\exp\,(M^*(t))}$ and

$$M^*(t) = \sup \log \frac{t^p}{M_p^*}$$
$$= \sup \log \frac{t^p}{\frac{M^p}{p!}}\ (because\ M_p^* = \frac{M_p}{p!})$$
$$= \sup \log \frac{p!\,t^p}{M_p}$$

Let the sequence M_p satisfies the condition (M.1), then the condition of strong non- quasi analyticity implies that for every $L > 0$ there exists a constant $C > 0$ such that $! \leq CL^p M_p, p \in \mathbb{N}_0 = \{0, 1, 2, 3, 4, \}$.

Therefore

$$M^*(t) = \sup \log \frac{p!\,t^p}{M_p} \leq \sup \log \frac{CL^p t^p}{M_p}$$
$$= \sup CL^p t^p.$$

Since $M_p^* p! \sim M_p$, we have $M_p^* p! \leq M_p^* CL^p M_p \leq CL^p M_p$. Therefore $M_p^* \leq L^p M_p$. Hence $M_p^* \simeq M_p$.

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